

# Metric-Like Formalism for Matter Fields Coupled to 3D Higher Spin Gravity

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## Abstract

Action integral for a matter system composed of 0- and 2-forms,  $C$  and  $B_{\mu\nu}$ , topologically coupled to 3D spin-3 gravity is considered in the frame-like formalism, and then the spin connection is eliminated by solving the eq of motion for the total action. It is shown that in the resulting metric-like formalism,  $(BC)^2$  interaction terms are induced because of the torsion. The world-volume components of the matter field,  $C^0$ ,  $C^\mu$  and  $C^{(\mu\nu)}$ , are introduced by contracting the local-frame index of  $C$  with those of the inverse vielbeins,  $E_a^\mu$  and  $E_a^{(\mu\nu)}$ , which were defined by the present authors in arXiv:1209.0894 [hep-th].

3D higher spin gravity theory contains various metric-like fields. These metric-like fields, as well as the new connections and the generalized curvature tensors, introduced in the above mentioned paper, are explicitly expressed in terms of the metric  $g_{\mu\nu}$  and the spin-3 field  $\phi_{\mu\nu\lambda}$  by means of the  $\phi$ -expansion. The matter action is re-expressed in terms of  $g_{\mu\nu}$ ,  $\phi_{\mu\nu\rho}$  and the covariant derivatives for spin-3 geometry. It is found that the action in the metric-like formalism is invariant under diffeomorphisms, just because the action in the frame-like formalism is topological: the  $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$  gauge symmetry in the matter-coupled theory does not contain true diffeomorphisms. This is due to the extra terms in the transformation rules which depend on the matter fields. Spin-3 gauge transformation is extended to the matter fields.

The action integral for the pure spin-3 gravity in the metric-like formalism up to  $\mathcal{O}(\phi^2)$ , obtained before in the literature, is rederived.

# 1 Introduction

In these several decades, study of higher-spin gauge theories has made steady progress. In the frame-like approach, Vasiliev proposed non-linear equations of motion for infinite tower of higher-spin gauge fields.[1][2][3][4] Although its description based on an action principle is still under investigation, it was conjectured that the higher-spin gravity in 3 dimensions is dual to the 2D W minimal CFT models,[5][6][8][7][9] and this duality (correspondence) has been studied in the version of the model with linearized scalar fields.[10][11][12][13][14]

It was also noticed that in 3 dimensions great simplifications occur.[15][16] The higher-spin fields can be truncated to only those with spin  $s \leq N$  and the theory with negative cosmological constant in the frame-like approach can be defined in terms of the  $SL(N, R) \times SL(N, R)$  Chern-Simons action. Various black hole solutions were found and their properties were studied.[17][18][21][19][11][20][22][23][24] In this frame-like approach the gravity and the higher-spin fields are described in terms of the vielbein  $e_\mu^a$  ( $a = 1, \dots, N^2 - 1$ ) and the spin connection  $\omega_\mu^a$  and the action integral is first order in the derivatives of these fields.

In gravity theories, there also exists a metric-like approach. In this approach the fields are metric tensor  $g_{\mu\nu}$  and higher-spin gauge fields, and an action for massless higher-spin fields was proposed by Fronsdal.[25] Correlation functions on the boundary CFT are studied by using holographic renormalization.[26] Cubic interaction vertices were also constructed.[27] In this approach the action is second order in the derivatives. It is more suitable for understanding of geometric properties.

In 3D spin-3 gravity such an action was constructed in [28] by perturbation in powers of the spin-3 field  $\phi_{\mu\nu\rho}$  up to  $\mathcal{O}(\phi^2)$ . In [29] the present authors eliminated the spin connection from the  $SL(3, R) \times SL(3, R)$  Chern-Simons theory by solving the torsion-free condition and then substituting the solution into the action integral. In this way we obtained an action integral which is quadratic in the derivatives of fields without using the perturbative methods. For this purpose we introduced a subsidiary vielbein  $e_{(\mu\nu)}^a$ , which is expressed in terms of the ordinary vielbein  $e_\mu^a$ , in order to define inverse vielbeins,  $E_a^\mu$ ,  $E_a^{(\mu\nu)}$ . This allowed us to solve the torsion-free conditions. We defined generalizations of the Christoffel symbols and curvature tensors. However, the action still contains metric-like fields expressed in terms of the vielbeins and the structure constants of the Lie algebra. In this sense, although the theory is in the second-order formalism, it is not a complete metric-like formalism. On the other hand, the existence of the second-order formalism is explicitly demonstrated. Advantage of our formalism is that this can be established to all orders in  $\phi$ . To express the action only in terms of the metric-like fields, it is necessary to express the vielbein in

terms of the metric and the spin-3 field, and substitute the result into the action integral. This can be performed only by perturbation in powers of the spin-3 field. This is one of the purposes of this paper. We will express  $\Gamma_{\mu N}^M$ ,  $R^M_{N\mu\nu}$  and other quantities in terms of  $g_{\mu\nu}$  and  $\phi_{\mu\nu\lambda}$  explicitly, and then express the action integral for spin-3 gravity in terms of the metric-like fields, and justify our formalism. The action for the spin-3 field turned out to agree with the result of [28].

Now, although 3D higher-spin gauge theory can be formulated in terms of the Chern-Simons theory, this is just a ‘pure spin-3 gravity’ theory. It is desirable to include matter fields. Actually, there must be ‘scalar fields’ in Vasiliev theory. It is, however, a very difficult task to construct action integrals for matter fields interacting with higher-spin gravity fields in an invariant manner in the metric-like approach. To our knowledge, this has never been done in the literature. In the second half of this paper, we will extend the 3D spin-3 gravity theory by topologically coupling matter fields. Matter fields are a 0-form  $C$  and a 2-form  $B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$ .

These fields are  $3 \times 3$  matrices and have local frame indices:  $C^a$ ,  $C^0$ ,  $B_{a\mu\nu}$ ,  $B_{0\mu\nu}$ . ( $a = 1, 2, \dots, 8$ ) This is a frame-like approach. By contracting these indices with those of the vielbeins  $e_\mu^a$ ,  $e_{(\mu\nu)}^a$ , or their inverse  $E_a^\mu$ ,  $E_a^{(\mu\nu)}$ , we obtain matter fields with world-volume indices in the metric-like approach:  $C^0$ ,  $C^\mu = E_a^\mu C^a$ ,  $C^{(\mu\nu)} = E_a^{(\mu\nu)} C^a$ , *etc.* The  $C$  field is not just a scalar field, but turned into a set of scalar, vector and tensor fields. Similarly for  $B$ .

In order to construct the spin-3 gravity theory interacting with the matter fields in the metric-like formalism, we need to eliminate the spin connection  $\omega_\mu^a$  by using some of the equations of motion. There are two options: one is to use the spin connection  $\omega_\mu^a(e)$  obtained by solving the equation of motion for the Chern-Simons theory representing the pure spin-3 gravity. The other is to solve the equation of motion for the total action to obtain  $\tilde{\omega}_\mu^a(e, B, C)$ . Firstly, we tried the first choice, and found that this leads to an action which is not invariant under spin-3 transformation. Since it does not seem simple to modify the action and transformation rules to recover the symmetry, we will adopt the second one. Then the equation of motion with respect to the spin connection is different from the one for the pure spin-3 gravity: a torsion tensor appears. Accordingly, the Christoffel-like connections and the spin connection must be replaced by new ones. We will show that the action integral for the pure spin-3 gravity sector and that for the matter sector get extra interaction terms, which are quadratic in the torsion tensor and of quartic order of matter

fields,  $(BC)^2$ . The action for the gravity sector is given by

$$S_{\text{CS}} = \frac{k}{48\pi\ell} \int d^3x \left( -\epsilon^{\mu\nu\lambda} F_{\mu M}{}^N R^M{}_{N\nu\lambda} + \frac{24}{\ell^2} \tilde{e} - 2\epsilon^{\mu\nu\lambda} F_{\mu M}{}^N \Delta \Gamma_{\nu K}^M \Delta \Gamma_{\lambda N}^K \right) \quad (1.1)$$

$\tilde{e}$  is a generalized cosmological term (3.20).  $F_{\mu M}{}^N$  is a metric-like tensor defined in (3.37) and appendix D. The first and the second terms can be rewritten as a sum of Einstein-Hilbert action and Fronsdal's spin-3 gauge action. The matter action in the metric-like formulation is given by

$$S_{\text{matter}} = \int d^3x \sqrt{-g} \left[ B_N{}^\lambda \left( \nabla_\lambda C^N + \frac{1}{\ell} K^N{}_{M\lambda} C^M \right) + \frac{2}{\ell} B_\lambda{}^\lambda C^0 \right. \\ \left. + B_0{}^\lambda \left( \partial_\lambda C^0 + \frac{4}{3\ell} C_\lambda \right) + B_N{}^\lambda C^M \Delta \Gamma_{\lambda M}^N \right] \quad (1.2)$$

Here  $\nabla_\lambda$  is a covariant derivative associated with the connections  $\Gamma_{\mu N}^M$ .  $B_N{}^\lambda$  and  $B_0{}^\lambda$  are defined in (4.36)-(4.37), and another metric-like tensor  $K^N{}_{M\lambda}$  is given in (4.39) and appendix B.  $\Delta \Gamma_{\lambda M}^N$ , (4.23), is a shift of the connection  $\Gamma_{\lambda M}^N$  due to the torsion. Those terms which include  $\Delta \Gamma$  yield fourth order interactions of the form  $(BC)^2$ . Under spin-3 transformation, fields  $C^0$ ,  $C^\mu$  and  $C^{(\mu\nu)}$  transform into each other, and the transformation rule will be obtained explicitly.

The novel feature of the spin-3 gravity with matter coupling is that local translation of the metric  $g_{\mu\nu}$  and the spin-3 gauge field  $\phi_{\mu\nu\lambda}$  contain terms which depend on the matter fields  $C$ ,  $B$  through the torsion tensor. Therefore the local translation does not contain the true diffeomorphism. However, our spin-3 gravity with matter coupling is still invariant under diffeomorphism, because the action in the frame-like approach is a topological one. Therefore the symmetry of the spin-3 gravity theory is enhanced by the matter coupling. We also find that the transformation rules of the matters,  $C$  and  $B$ , become non-linear in the matter fields.

This paper is organized as follows. In sec.2 our second order formalism will be reviewed briefly. In sec.3 various metric-like fields will be expressed in terms of the metric  $g_{\mu\nu}$  and the spin-3 field  $\phi_{\mu\nu\lambda}$  by using perturbation in  $\phi$ . The four kinds of the generalized Christoffel connection  $\Gamma_{\mu N}^M$ , the generalized curvature tensor  $R^M{}_{N\mu\nu}$  and the action integral for the pure spin-3 gravity, are then expressed in terms of these fields. The transformation properties of  $g_{\mu\nu}$  and  $\phi_{\mu\nu\lambda}$  are then studied. In sec.4 an action integral for matter fields coupled to the spin-3 gravity is explicitly written down. The action integral is then converted into a metric-like form by contracting the indices of the matter fields with those of the vielbeins. Transformation rules of the matter fields under spin-3 transformation will be worked out explicitly. Sec.5 is a summary. In appendices A-G, some formulae are presented. We performed various computations by using *xAct* packages for Mathematica[30].

## 2 Brief Review of the Formalism

In [29] we defined a subsidiary field  $e_{(\mu\nu)}^a$  in terms of the vielbein field  $e_\mu^a$ .

$$e_{(\mu\nu)}^a = \frac{1}{2} d^a_{bc} e_\mu^b e_\nu^c - \frac{1}{6} g_{\mu\nu} g^{\lambda\rho} d^a_{bc} e_\lambda^b e_\rho^c \quad (2.1)$$

This is symmetric under interchange of  $\mu$  and  $\nu$ . The second term on the righthand side ensures that  $e_{(\mu\nu)}^a$  is traceless:  $g^{\mu\nu} e_{(\mu\nu)}^a = 0$  so that  $e_{(\mu\nu)}^a$  has five independent components with respect to the indices  $\mu, \nu$ . In spin-3 gravity the index  $a$  is associated with  $SL(3, R)$  and runs over  $a = 1, 2, \dots, 8$ . So the set of generalized vielbeins,  $(e_\mu^a$  and  $e_{(\mu\nu)}^a)$ , makes up an  $8 \times 8$  matrix.

This allows us to define the inverse vielbeins,  $E_a^\mu$  and  $E_a^{(\mu\nu)}$ , by the relations,

$$\begin{aligned} E_a^\mu e_\nu^a &= \delta_\nu^\mu, & E_a^\mu e_{(\nu\lambda)}^a &= 0, \\ E_a^{(\mu\nu)} e_\lambda^a &= 0, & E_a^{(\mu\nu)} e_{(\lambda\rho)}^a &= \delta_\lambda^\mu \delta_\rho^\nu + \delta_\lambda^\nu \delta_\rho^\mu - \frac{2}{3} g_{\lambda\rho} g^{\mu\nu}, \end{aligned} \quad (2.2)$$

and then solve the torsion-free condition  $\partial_\mu e_\nu^a - \partial_\nu e_\mu^a + f^a_{bc} \omega_\mu^b e_\nu^c - f^a_{bc} \omega_\nu^b e_\mu^c = 0$  and express the spin connection  $\omega_\mu^a$  in terms of the vielbeins. The result is

$$\omega_\mu^a = \omega_\mu^a(e) \equiv \frac{1}{12} f^{ab}{}_c E_b^\lambda \nabla_\mu e_\lambda^c + \frac{1}{24} f^{ab}{}_c E_b^{(\lambda\rho)} \nabla_\mu e_{(\lambda\rho)}^c. \quad (2.3)$$

Here  $\nabla_\mu$  is a new covariant derivative. For this definition, we need to introduce some notation for indices. Let  $M, N, \dots$  denote a set of two types of indices,  $\mu$  and  $(\nu, \lambda)$ . Then covariant derivatives of tensors,  $V_M, V^M$ , with this type of indices are defined by

$$\nabla_\mu V_M = \partial_\mu V_M - \Gamma_{\mu M}^N V_N = \partial_\mu V_M - \Gamma_{\mu M}^\nu V_\nu - \frac{1}{2} \Gamma_{\mu M}^{(\nu\lambda)} V_{(\nu\lambda)}, \quad (2.4)$$

$$\nabla_\mu V^M = \partial_\mu V^M + \Gamma_{\mu N}^M V^N = \partial_\mu V^M + \Gamma_{\mu\nu}^M V^\nu + \frac{1}{2} \Gamma_{\mu(\nu\lambda)}^M V^{(\nu\lambda)}. \quad (2.5)$$

Note that a factor  $\frac{1}{2}$  is associated to the summation over the pair of indices  $(\mu\nu)$ . We will use this summation convention throughout this paper. There are four types of connections  $\Gamma_{\mu N}^M$  according to the types of  $M$  and  $N$ . These are generalizations of the Christoffel symbol in the ordinary gravity.<sup>1</sup> In [29] the expression for these connections are determined in terms of the metric-like quantities in such a way that the full covariant derivatives of the generalized vielbeins vanish:  $D_\mu e_\nu^a = \nabla_\mu e_\nu^a + f^a_{bc} \omega_\mu^b e_\nu^c = 0$  and  $D_\mu e_{(\nu\lambda)}^a = \nabla_\mu e_{(\nu\lambda)}^a + f^a_{bc} \omega_\mu^b e_{(\nu\lambda)}^c = 0$ . The (extended) metric compatible with the covariant derivatives  $\nabla_\mu$  is given by

$$G_{MN} = e_M^a e_{aN}. \quad (2.6)$$

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<sup>1</sup>A generalization of Christoffel symbol in higher-spin gauge theories was once considered in [36]. The direction of the generalization is, however, different from ours.

This is decomposed into four blocks and three of them are related to the metric and the spin-3 field: the first block is the ordinary metric,  $G_{\mu\nu} = g_{\mu\nu}$ . Off-diagonal blocks are  $G_{\mu(\nu\rho)} = G_{(\nu\rho)\mu} = \phi_{\mu\nu\rho} - \frac{1}{3} g_{\nu\rho} g^{\lambda\sigma} \phi_{\lambda\sigma\mu}$ , where  $\phi_{\mu\nu\rho}$  is the spin-3 field. The last one  $G_{(\mu\nu)(\lambda\rho)}$  is new, but in principle can be expressed in terms of  $g_{\mu\nu}$  and  $\phi_{\mu\nu\lambda}$ , as is displayed in appendix A. However, the covariant derivative  $\nabla_\mu$  defined above mixes the two types of indices,  $\mu$  and  $(\mu\nu)$ , and the last component  $G_{(\mu\nu)(\lambda\rho)}$  is also important. The perturbative expansions of  $G_{MN}$  and  $G^{MN}$  in powers of  $\phi$  are given in appendix A.

In order to distinguish the ordinary Christoffel symbol from the above new connections  $\Gamma_{\mu N}^M$ , the former will be denoted as  $\hat{\Gamma}_{\nu\lambda}^\mu$  throughout this paper. The covariant derivatives and the curvature tensor associated with the Christoffel symbol will be denoted as  $\hat{\nabla}_\mu$  and  $\hat{R}^\lambda_{\rho\mu\nu}$ , respectively.

3D spin-3 gravity is defined by  $SL(3, R) \times SL(3, R)$  Chern-Simons action and the field variables are the vielbein  $e_\mu^a$  and the spin connection  $\omega_\mu^a$ . This is a first-order formalism. By substituting  $\omega_\mu^a(e)$  in (2.3) into  $\omega_\mu^a$  in the Chern-Simons action, an action integral in the second-order formalism was obtained in [29]. In the next section we will express the action integral and several geometric quantities only in terms of the metric and the spin-3 field by using perturbative expansions in  $\phi$ .

### 3 Vielbein in terms of metric-like fields

As was explained in [29], there are various metric-like fields in spin-3 gravity. Among them the metric field  $g_{\mu\nu}$  and the spin-3 field  $\phi_{\mu\nu\lambda}$  are important because the others are assumed to be expressible in terms of these. They are define by<sup>2</sup>

$$g_{\mu\nu} = \frac{1}{2} \text{tr } e_\mu e_\nu = h_{ab} e_\mu^a e_\nu^b, \quad (3.1)$$

$$\phi_{\mu\nu\lambda} = \frac{1}{4} \text{tr } e_\mu \{e_\nu, e_\lambda\} = \frac{1}{2} d_{abc} e_\mu^a e_\nu^b e_\lambda^c \quad (3.2)$$

Here  $\{ , \}$  is an anti-commutator. Other metric-like fields are similarly defined in terms of traces of products of the vielbeins, so if the above relations were solved for  $e_\mu$ , all the metric-like fields would be expressed in terms of  $g_{\mu\nu}$  and the spin-3 field  $\phi_{\mu\nu\lambda}$ . This is what we are up to. The vielbein  $e_\mu^a$  has 24 components, and among them 8 of those are gauge degrees of freedom for local 'Lorentz rotations'. Up to these gauge transformations, the vielbein is expected to be obtained uniquely.

By using the Killing metric  $h_{ab}$ <sup>3</sup> the explicit form of relation (3.1) reads

$$g_{\mu\nu} = e_\mu^2 e_\nu^2 - 2(e_\mu^1 e_\nu^3 + e_\mu^3 e_\nu^1) + 8(e_\mu^4 e_\nu^8 + e_\mu^8 e_\nu^4) - 2(e_\mu^5 e_\nu^7 + e_\mu^7 e_\nu^5) + \frac{4}{3} e_\mu^6 e_\nu^6 \quad (3.3)$$

<sup>2</sup>For the definitions of the basis  $t_a$  of  $sl(3, R)$ , see appendix A of [29].

<sup>3</sup>See appendix A of [29] for our conventions.

Similar, but more involved eqs for  $\phi$  can also be written down explicitly using the symmetric structure constant  $d_{abc}$ . The indices  $\mu, \nu, \dots$  run over  $r, t, \phi$ .

At present, this attempt can be fulfilled only perturbatively: we must assume that  $\phi_{\mu\nu\lambda}$  is small, and resort to expansions in powers of  $\phi$ . We will use the following gauge fixing conditions for local frame rotations.

$$e_r^a = 0, \quad (a \neq 2, 6) \quad (3.4)$$

The remaining two are given by

$$e_t^1 = e_t^3, \quad e_t^7 e_\phi^1 + e_t^5 e_\phi^3 = 0. \quad (3.5)$$

First, the eqs for  $g_{rr}$  and  $\phi_{rrr}$ ,

$$g_{rr} = (e_r^2)^2 + \frac{4}{3} (e_r^6)^2, \quad (3.6)$$

$$\phi_{rrr} = -\frac{8}{9} (e_r^6)^3 + 2 (e_r^2)^2 e_r^6, \quad (3.7)$$

are combined into a cubic eq for  $e_r^6$

$$(e_r^6)^3 - \frac{9}{16} g_{rr} e_r^6 + \frac{9}{32} \phi_{rrr} = 0 \quad (3.8)$$

This eq has one or three real roots according to the sign of the discriminant  $D = (81/4096) \phi_{rrr}^2 - (27/256) g_{rr}^3$ : for  $D < 0$  there are three real roots.

$$e_r^6 = \omega^k \left( -\frac{9}{64} \phi_{rrr} + i \sqrt{-D} \right)^{1/3} + \omega^{3-k} \left( -\frac{9}{64} \phi_{rrr} - i \sqrt{-D} \right)^{1/3} \quad (k = 0, 1, 2) \quad (3.9)$$

Here  $\omega = e^{2\pi i/3}$ , and  $\sqrt{-D}$  denotes the positive root. For  $D \geq 0$  there is only a single one.

$$e_r^6 = \left( -\frac{9}{64} \phi_{rrr} + \sqrt{D} \right)^{1/3} + \left( -\frac{9}{64} \phi_{rrr} - \sqrt{D} \right)^{1/3} \quad (3.10)$$

Assuming that  $g_{rr} > 0$ ,<sup>4</sup>  $D$  turns out negative for small  $|\phi_{rrr}|$ , and there are three solutions for the inversion problem. For  $\phi = 0$ , the solutions are  $e_r^6 = 0, \pm \frac{3}{4} \sqrt{g_{rr}}$ . Because  $\phi = 0$  corresponds to the ordinary spin-2 gravity, in what follows, we will choose the branch which reduces to  $e_r^6 = 0$  at  $\phi = 0$ . This solution is smoothly connected to the one for  $D \geq 0$  and large  $|\phi_{rrr}|$ . Then the small  $\phi$  expansion for  $e_r^6$  is given by

$$e_r^6 = \frac{1}{2} (g_{rr})^{-1} \phi_{rrr} + \frac{2}{9} (g_{rr})^{-4} (\phi_{rrr})^3 + \dots \quad (3.11)$$

In turn,  $e_r^2$  is determined by solving (3.6).

$$e_r^2 = \sqrt{g_{rr} - \frac{4}{3} (e_r^6)^2} = \sqrt{g_{rr}} + \dots \quad (3.12)$$

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<sup>4</sup>Throughout this paper the world-volume indices take values  $\mu, \nu, \dots = r, t, \phi$ , and the signature is  $(+, -, +)$ .

The above eq shows that there is an upper bound for  $|\phi_{rrr}|$ , given the value of  $g_{rr}$ : (3.8) shows that  $\phi_{rrr}$  grows as  $(e_r^6)^3$  for large  $e_r^6$ .

The eqs for  $g_{rt}$ ,  $g_{r\phi}$ ,  $\phi_{rrt}$  and  $\phi_{rr\phi}$  determine  $e_t^6$ ,  $e_\phi^6$ ,  $e_t^2$  and  $e_\phi^2$ . Results are

$$e_t^6 = \frac{\frac{3}{2}\phi_{rrt} - 2g_{rt}e_r^6}{g_{rr} - \frac{16}{3}(e_r^6)^2}, \quad (3.13)$$

$$e_r^2 = \frac{(g_{rr} - \frac{8}{3}(e_r^6)^2)g_{rt} - 2\phi_{rrt}e_r^6}{g_{rr} - \frac{16}{3}(e_r^6)^2}, \quad (3.14)$$

$$e_\phi^6 = \frac{\frac{3}{2}\phi_{rr\phi} - 2g_{r\phi}e_r^6}{g_{rr} - \frac{16}{3}(e_r^6)^2}, \quad (3.15)$$

$$e_\phi^2 = \frac{(g_{rr} - \frac{8}{3}(e_r^6)^2)g_{r\phi} - 2\phi_{rr\phi}e_r^6}{(g_{rr} - \frac{16}{3}(e_r^6)^2)e_r^2}. \quad (3.16)$$

The solution (3.11) is to be substituted into the above.

The remaining eqs for  $g_{tt}$ ,  $g_{t\phi}$ ,  $g_{\phi\phi}$ ,  $\phi_{rtt}$ ,  $\phi_{rt\phi}$ ,  $\phi_{r\phi\phi}$ ,  $\phi_{ttt}$ ,  $\phi_{tt\phi}$ ,  $\phi_{t\phi\phi}$  and  $\phi_{\phi\phi\phi}$  provide a set of coupled eqs that can at best be solved only by perturbations in powers of  $\phi$ . The redundancy in these eqs can be removed by the gauge fixing (3.5). To leading order  $\mathcal{O}(\phi^0)$  we have

$$e^1 = \frac{1}{2}\sqrt{\frac{-g}{g_{rr}}}g^{\phi\phi}dt + \frac{1}{2\sqrt{g_{rr}g^{\phi\phi}}}(-\sqrt{-g}g^{t\phi} - \sqrt{g_{rr}})d\phi, \quad (3.17)$$

$$e^2 = \sqrt{g_{rr}}dr + \frac{1}{\sqrt{g_{rr}}}g_{rt}dt + \frac{1}{\sqrt{g_{rr}}}g_{r\phi}d\phi, \quad (3.18)$$

$$e^3 = \frac{1}{2}\sqrt{\frac{-g}{g_{rr}}}g^{\phi\phi}dt + \frac{1}{2\sqrt{g_{rr}g^{\phi\phi}}}(-\sqrt{-g}g^{t\phi} + \sqrt{g_{rr}})d\phi. \quad (3.19)$$

Here  $g^{\mu\nu}$  is the inverse of  $g_{\mu\nu}$  and  $g$  is the determinant of  $g_{\mu\nu}$ . Other vielbeins  $e^4, \dots, e^8$  begin with  $\mathcal{O}(\phi^1)$ . In order to compute the action integral to the first non-trivial order, and in terms of the metric-like fields, we computed  $e^1, e^2, e^3$  up to  $\mathcal{O}(\phi^2)$  and  $e^4, \dots, e^8$  up to  $\mathcal{O}(\phi^3)$ .

### 3.1 Connections and Curvature Tensor

By using the above results we can compute all quantities necessary for writing down the action integral. These include the generalized cosmological term.<sup>5</sup>

$$\tilde{e} = \frac{1}{6}\epsilon^{\mu\nu\lambda}f_{abc}e_\mu^ae_\nu^be_\lambda^c = \sqrt{-g}\left(1 + \frac{1}{2}\phi_\mu\phi^\mu - \frac{1}{3}\phi_{\mu\nu\lambda}\phi^{\mu\nu\lambda} + \mathcal{O}(\phi^4)\right) \quad (3.20)$$

Here  $\epsilon^{\mu\nu\lambda}$  is a unit completely anti-symmetric symbol with  $\epsilon^{rt\phi} = +1$ .

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<sup>5</sup>  $\phi_\mu = \phi_{\mu\nu\lambda}g^{\nu\lambda}$ . Indices are raised and lowered by  $g^{\mu\nu}$  and  $g_{\mu\nu}$ , except for  $\xi^M$ ,  $\xi_M$  and  $C^M$ ,  $C_M$  in sec.4. In the latter case,  $G^{MN}$  and  $G_{MN}$  play the role of interchanging the indices.



To compute Christoffel-like connections  $\Gamma_{\mu N}^M$ , it is necessary to define tensor  $M_{(\mu\nu)(\lambda\rho)}$  and its inverse,  $J^{(\mu\nu)(\lambda\rho)}$ . These are defined[29] as

$$M_{(\mu\nu)(\lambda\rho)} = G_{(\mu\nu)(\lambda\rho)} - G_{(\mu\nu)\alpha} g^{\alpha\beta} G_{\beta(\lambda\rho)}, \quad (3.21)$$

$$\frac{1}{2} J^{(\mu\nu)(\sigma\kappa)} M_{(\sigma\kappa)(\lambda\rho)} = P_{\lambda\rho}^{\mu\nu} \equiv \delta_\lambda^\mu \delta_\rho^\nu + \delta_\rho^\mu \delta_\lambda^\nu - \frac{2}{3} g^{\mu\nu} g_{\lambda\rho} \quad (3.22)$$

Here  $P_{\mu\nu}^{\lambda\rho}$  is a projector onto a space of symmetric traceless tensors.<sup>6</sup> These have the expansion,  $M = M^{(0)} + M^{(2)} + \dots$ , where  $M^{(n)} = \mathcal{O}(\phi^n)$ , and similarly for  $J$ . The 0-th terms are given by

$$M_{(\mu\nu)(\lambda\rho)}^{(0)} = \frac{1}{4} g_{\mu\sigma} g_{\nu\kappa} P_{\lambda\rho}^{\sigma\kappa}, \quad (3.23)$$

$$J^{(0)(\mu\nu)(\lambda\rho)} = 4 g^{\sigma\lambda} g^{\kappa\rho} P_{\sigma\kappa}^{\mu\nu} \quad (3.24)$$

One of the four connections,  $\Gamma_{\mu\nu}^{(\lambda\rho)}$ , is given by

$$\Gamma_{\mu\nu}^{(\lambda\rho)} = \frac{1}{6} (\hat{\nabla}_\mu \Phi_{\nu\kappa\sigma} + \hat{\nabla}_\nu \Phi_{\mu\kappa\sigma} - \hat{\nabla}_\kappa \Phi_{\mu\nu\sigma}) J^{(\kappa\sigma)(\lambda\rho)} + \frac{1}{4} S_{\mu\nu,\kappa\sigma} J^{(\kappa\sigma)(\lambda\rho)}. \quad (3.25)$$

Here  $\Phi$  is the traceless part of  $\phi$ :  $\Phi_{\mu\nu\lambda} = \phi_{\mu\nu\lambda} - \frac{1}{5} (g_{\mu\nu} \phi_\lambda + g_{\nu\lambda} \phi_\mu + g_{\lambda\mu} \phi_\nu)$ .  $S_{\mu\nu,\kappa\sigma}$  is a tensor obtained by solving a few algebraic equations<sup>7</sup> and the formal solution was presented in appendix D of [29], where  $S_{\mu\nu,\kappa\sigma}$  is given as an infinite sum of  $S_{\mu\nu,\kappa\sigma}^{(n)}$ , which is  $\mathcal{O}(\phi^{2n+1})$ . Up to  $\mathcal{O}(\phi^2)$ , it is given by

$$\begin{aligned} S_{\mu\nu,\lambda\rho} = S_{\mu\nu,\lambda\rho}^{(0)} &= -\frac{3}{5} g_{\mu\nu} (\mathcal{A}_{\lambda\rho} + \mathcal{A}_{\rho\lambda}) - \frac{2}{5} g_{\lambda\rho} (\mathcal{A}_{\mu\nu} + \mathcal{A}_{\nu\mu}) \\ &+ \frac{3}{5} g_{\mu\nu} g_{\lambda\rho} \mathcal{A}_\alpha^\alpha - \frac{3}{10} (g_{\mu\lambda} g_{\nu\rho} + g_{\mu\rho} g_{\nu\lambda}) \mathcal{A}_\alpha^\alpha \\ &+ \frac{1}{5} g_{\mu\lambda} (2 \mathcal{A}_{\nu\rho} + \mathcal{A}_{\rho\nu}) + \frac{1}{5} g_{\nu\lambda} (2 \mathcal{A}_{\mu\rho} + \mathcal{A}_{\rho\mu}) \\ &+ \frac{1}{5} g_{\mu\rho} (2 \mathcal{A}_{\nu\lambda} + \mathcal{A}_{\lambda\nu}) + \frac{1}{5} g_{\nu\rho} (2 \mathcal{A}_{\mu\lambda} + \mathcal{A}_{\lambda\mu}) + \mathcal{O}(\phi^3), \end{aligned} \quad (3.26)$$

where  $\mathcal{A}_{\mu\nu}$  is given by

$$\begin{aligned} \mathcal{A}_{\mu\nu} &= \hat{\nabla}_\mu \phi_\nu - \frac{5}{9} \hat{\nabla}^\lambda \Phi_{\mu\nu\lambda} + \mathcal{O}(\phi^3) \\ &= \frac{10}{9} \hat{\nabla}_\mu \phi_\nu + \frac{1}{9} \hat{\nabla}_\nu \phi_\mu - \frac{5}{9} \hat{\nabla}_\rho \phi_{\mu\nu}{}^\rho + \frac{1}{9} g_{\mu\nu} \hat{\nabla}_\lambda \phi^\lambda + \mathcal{O}(\phi^3). \end{aligned} \quad (3.27)$$

In eq (D.2) of [29], which defines  $\mathcal{A}_{\mu\nu}$ , there also appears a term containing  $W_{\sigma\kappa}$ . This quantity, however, can be shown to be  $\mathcal{O}(\phi^2)$  and does not contribute here.

By using (3.28), (3.24) and (3.26), we obtain

$$\begin{aligned} \Gamma_{\mu\nu}^{(\lambda\rho)} &= -\frac{4}{3} \hat{\nabla}^{(\lambda} \phi^{\rho)}_{\mu\nu} - \frac{8}{3} g_{\mu\nu} \hat{\nabla}^{(\lambda} \phi^{\rho)} + \frac{8}{3} \delta_{(\mu}^{(\lambda} \hat{\nabla}_{\nu)} \phi^{\rho)} + \frac{8}{3} \hat{\nabla}_{(\mu} \phi^{\lambda\rho}{}_{\nu)} \\ &+ \frac{8}{3} \delta_{(\mu}^{(\lambda} \hat{\nabla}_{\nu)} \phi^{\rho)} - \frac{8}{3} g^{\lambda\rho} \hat{\nabla}_{(\mu} \phi_{\nu)} + \frac{4}{3} g_{\mu\nu} \hat{\nabla}_\sigma \phi^{\lambda\rho\sigma} - \frac{8}{3} \delta_{(\mu}^{(\lambda} \hat{\nabla}_{|\sigma|} \phi^{\rho)}{}_{\nu)}{}^\sigma \\ &+ \frac{4}{3} g^{\lambda\rho} \hat{\nabla}_\sigma \phi_{\mu\nu}{}^\sigma - \frac{2}{3} \delta_{\mu}^{(\lambda} \delta_{\nu)}^{\rho)} \hat{\nabla}_\sigma \phi^\sigma + \frac{2}{3} g^{\lambda\rho} g_{\mu\nu} \hat{\nabla}_\sigma \phi^\sigma + \mathcal{O}(\phi^3). \end{aligned} \quad (3.28)$$

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<sup>6</sup>  $\frac{1}{2} P_{\kappa\sigma}^{\mu\nu} P_{\lambda\rho}^{\kappa\sigma} = P_{\lambda\rho}^{\mu\nu}$   
<sup>7</sup> Eqs (3.30) and (3.32) in [29]

Indices between parentheses are meant to be completely symmetrized, and dividing by the number of terms that are needed for symmetrization is understood.

Then  $\Gamma_{\mu\nu}^\lambda$  is obtained by using the formula  $\Gamma_{\mu\nu}^\lambda = \hat{\Gamma}_{\mu\nu}^\lambda - \frac{1}{2} \Gamma_{\mu\nu}^{(\sigma\kappa)} \phi_{\sigma\kappa}^\lambda$ .<sup>8</sup>

$$\begin{aligned} \Gamma_{\mu\nu}^\lambda &= \hat{\Gamma}_{\mu\nu}^\lambda - \frac{4}{3} \phi^{\lambda\rho\sigma} \hat{\nabla}_{(\mu} \phi_{\nu)\rho\sigma} + \frac{4}{3} \phi^\lambda \hat{\nabla}_{(\mu} \phi_{\nu)} - \frac{4}{3} \phi^\lambda{}^\rho \hat{\nabla}_\rho \phi_\nu + \frac{1}{3} \phi^\lambda{}_{\mu\nu} \hat{\nabla}_\rho \phi^\rho \\ &\quad - \frac{4}{3} \phi^\lambda{}_{(\mu} \hat{\nabla}_{|\rho|} \phi_{\nu)} + \frac{2}{3} \phi^{\lambda\rho\sigma} \hat{\nabla}_\rho \phi_{\mu\nu\sigma} - \frac{2}{3} \phi^\lambda \hat{\nabla}_\rho \phi_{\mu\nu}{}^\rho + \frac{4}{3} \phi^{\lambda\rho}{}_{(\mu} \hat{\nabla}_{|\sigma|} \phi_{\nu)\rho}{}^\sigma \\ &\quad + \frac{4}{3} g_{\mu\nu} \phi^{\lambda\rho\sigma} \hat{\nabla}_\rho \phi_\sigma - \frac{2}{3} g_{\mu\nu} \phi^{\lambda\rho\sigma} \hat{\nabla}_\kappa \phi_{\rho\sigma}{}^\kappa - \frac{1}{3} g_{\mu\nu} \phi^\lambda \hat{\nabla}_\rho \phi^\rho + \mathcal{O}(\phi^4) \end{aligned} \quad (3.29)$$

The remaining two,  $\Gamma_{\mu(\nu\lambda)}^\rho$  and  $\Gamma_{\mu(\nu\lambda)}^{(\rho\sigma)}$  are obtained by using eqs (4.12) and (4.13) in [29]. For computing  $\Gamma_{\mu(\nu\lambda)}^\rho$  to the leading order, which is  $\mathcal{O}(\phi^1)$ , we need to evaluate a quantity,

$$\begin{aligned} K^{\rho}{}_{(\sigma\kappa)\lambda} &\equiv d^a{}_{bc} E_a^\rho e_{(\sigma\kappa)}^b e_\lambda^c \\ &= -\frac{1}{3} \delta_\lambda^\rho g_{\sigma\kappa} + \delta_{(\sigma}^\rho g_{\kappa)\lambda} + \mathcal{O}(\phi^2) \end{aligned} \quad (3.30)$$

and we obtain

$$\begin{aligned} \Gamma_{\mu(\nu\lambda)}^\rho &= -\frac{1}{3} \hat{\nabla}^\rho \phi_{\mu\nu\lambda} + \frac{1}{3} g_{\nu\lambda} \hat{\nabla}^\rho \phi_\mu - \frac{1}{3} g_{\mu(\nu} \hat{\nabla}^\rho \phi_{\lambda)} + \frac{2}{3} \hat{\nabla}_\mu \phi_{\nu\lambda}^\rho \\ &\quad - \frac{1}{3} \delta_{(\nu}^\rho \hat{\nabla}_{|\mu|} \phi_{\lambda)} + \frac{1}{3} \hat{\nabla}_{(\nu} \phi_{\lambda)\mu}^\rho - \frac{1}{3} g_{\mu(\nu} \hat{\nabla}_\lambda \phi_{\rho)}^\rho - \frac{1}{3} \delta_{(\nu}^\rho \hat{\nabla}_\lambda \phi_{\rho)}^\mu \\ &\quad + \frac{2}{3} \delta_\mu^\rho \hat{\nabla}_{(\nu} \phi_{\lambda)} - \frac{1}{3} g_{\nu\lambda} \hat{\nabla}_\sigma \phi^{\sigma\rho}{}_\mu + \frac{1}{3} g_{\mu(\nu} \hat{\nabla}_{|\sigma|} \phi^{\sigma\rho}{}_{\lambda)} + \frac{1}{3} \delta_{(\nu}^\rho \hat{\nabla}_{|\sigma|} \phi_{\lambda)\mu}{}^\sigma \\ &\quad - \frac{1}{3} \delta_\mu^\rho \hat{\nabla}_\sigma \phi_{\nu\lambda}{}^\sigma + \frac{1}{6} \delta_{(\nu}^\rho g_{\lambda)\mu} \hat{\nabla}_\sigma \phi^\sigma - \frac{1}{6} \delta_\mu^\rho g_{\nu\lambda} \hat{\nabla}_\sigma \phi^\sigma + \mathcal{O}(\phi^3). \end{aligned} \quad (3.31)$$

For computing  $\Gamma_{\mu(\nu\lambda)}^{(\rho\sigma)}$ , we also need to evaluate another quantity,

$$\begin{aligned} K^{(\rho\sigma)}{}_{(\kappa\eta)\lambda} &\equiv d^a{}_{bc} E_a^{(\rho\sigma)} e_{(\kappa\eta)}^b e_\lambda^c \\ &= -\frac{4}{3} g_{\lambda(\kappa} \phi^{\rho\sigma}{}_{\eta)} + \frac{4}{3} g_{\kappa\eta} \phi^{\rho\sigma}{}_\lambda + \frac{8}{3} \delta_\lambda^{(\rho} \phi^{\sigma)}{}_{\kappa\eta} - \frac{8}{3} \delta_{(\kappa}^{(\rho} \phi^{\sigma)}{}_{\eta)\lambda} - \frac{4}{3} g_{\lambda(\eta} \delta_{\kappa)}^{(\rho} \phi^{\sigma)} \\ &\quad - \frac{4}{3} \phi_{(\eta} \delta_{\kappa)}^{(\rho} \delta_{\lambda)}^{\sigma)} + \frac{4}{3} g^{\rho\sigma} g_{\lambda(\kappa} \phi_{\eta)} + \frac{4}{3} \delta_{(\kappa}^\rho \delta_{\eta)}^\sigma \phi_\lambda - \frac{8}{9} g^{\rho\sigma} g_{\kappa\eta} \phi_\lambda + \mathcal{O}(\phi^3) \end{aligned} \quad (3.32)$$

By using this tensor, we obtain

$$\begin{aligned} \Gamma_{\mu(\nu\lambda)}^{(\rho\sigma)} &= 4 \delta_{(\nu}^{(\rho} \Gamma_{|\mu|\lambda)}^{\sigma)} - \frac{4}{3} g^{\rho\sigma} g_{\kappa(\nu} \Gamma_{|\mu|\lambda)}^\kappa - \frac{2}{9} g^{\rho\sigma} g_{\nu\lambda} \phi_{\kappa\tau}{}^\eta \Gamma_{\mu\eta}^{(\kappa\tau)} + \frac{2}{3} g^{\eta(\rho} g_{\nu\lambda} \phi_{\kappa\tau}^{\sigma)} \Gamma_{\mu\eta}^{(\kappa\tau)} \\ &\quad - \frac{2}{9} g_{\nu\lambda} g^{\eta(\rho} \phi^{\sigma)} \Gamma_{\mu\eta}^{(\kappa\tau)} + \frac{1}{2} K^{(\rho\sigma)}{}_{(\kappa\tau)(\nu} \Gamma_{|\mu|\lambda)}^{(\kappa\tau)} \\ &\quad - \frac{1}{6} g_{\nu\lambda} g^{\alpha\eta} K^{(\rho\sigma)}{}_{(\kappa\tau)\eta} \Gamma_{\mu\alpha}^{(\kappa\tau)} + \mathcal{O}(\phi^4) \end{aligned} \quad (3.33)$$

Note that this expression contains other kinds of  $\Gamma$ 's and they must be substituted. The final result is complicated and will not be displayed here. This connection has a non-vanishing trace with respect to the lower paired indices:  $g^{\nu\lambda} \Gamma_{\mu(\nu\lambda)}^{(\rho\sigma)} = -2 \partial_\mu g^{\rho\sigma} - \frac{2}{3} g^{\rho\sigma} g^{\nu\lambda} \partial_\mu g_{\nu\lambda}$ . This

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<sup>8</sup>eq (3.17) in [29]

is because the definition of this connection,  $\Gamma_{\mu(\nu\lambda)}^{(\rho\sigma)} = E_a^{(\rho\sigma)} (\partial_\mu e_{(\nu\lambda)}^a - \nabla_\mu e_{(\nu\lambda)}^a)$  contains the derivative  $\partial_\mu$  of the vielbein. However, contrary to the expectation, the similarly defined connection,  $\Gamma_{\mu(\nu\lambda)}^\rho$ , does not have a trace,  $g^{\nu\lambda} \Gamma_{\mu(\nu\lambda)}^\rho = 0$ , as can be checked by using (3.31).

Finally, we turn to the generalized curvature tensor. This was defined in eq (6.11) of [29].

$$R^M{}_{N\mu\nu} \equiv \partial_\mu \Gamma_{\nu N}^M - \partial_\nu \Gamma_{\mu N}^M + \Gamma_{\mu K}^M \Gamma_{\nu N}^K - \Gamma_{\nu K}^M \Gamma_{\mu N}^K. \quad (3.34)$$

The components can be obtained by substituting the above results, and some of them are presented in appendix C. Here we make a brief comment. Firstly, to the leading order, the component  $R^\lambda{}_{\rho\mu\nu}$  agrees with the ordinary Riemann tensor in 3 dimensions, as it should.

$$R^\lambda{}_{\rho\mu\nu} = \hat{R}^\lambda{}_{\rho\mu\nu} + \mathcal{O}(\phi^2), \quad (3.35)$$

where  $\hat{R}^\lambda{}_{\rho\mu\nu} = \partial_\mu \hat{\Gamma}_{\nu\rho}^\lambda - \partial_\nu \hat{\Gamma}_{\mu\rho}^\lambda + \hat{\Gamma}_{\mu\kappa}^\lambda \hat{\Gamma}_{\nu\rho}^\kappa - \hat{\Gamma}_{\nu\kappa}^\lambda \hat{\Gamma}_{\mu\rho}^\kappa = \delta_\mu^\lambda \hat{R}_{\rho\nu} - \frac{1}{2} \hat{R} \delta_\mu^\lambda g_{\nu\rho} + \dots$ . The second-order terms are too complicated to present here.

Secondly, two types of components,  $R^{(\rho\sigma)}{}_{\lambda\mu\nu}$  and  $R^{(\rho\sigma)}{}_{(\lambda\kappa)\mu\nu}$ , turn out to contain terms proportional to the ordinary Christoffel symbols. Therefore, these components do not act as tensors under diffeomorphism. The reason can be traced to the derivative  $\partial_\mu$  on  $\Gamma_{\nu N}^M$  in (3.34). The traceless condition  $g_{\rho\sigma} R^{(\rho\sigma)}{}_{N\mu\nu} = 0$  is jeopardized by the derivative. However, those terms proportional to the Christoffels are all proportional to  $g^{\rho\sigma}$ , and by contracting these components by the projector  $P_{\rho\sigma}^{\alpha\beta}$  (3.22), we can obtain quantities covariant under diffeomorphisms. This means that more appropriate definition of the generalized curvature tensor may be the one, obtained by projecting out some terms by using (3.22), like  $[R^{(\rho\sigma)}{}_{\lambda\mu\nu}]_{\text{redefined}} = \frac{1}{2} P_{\alpha\beta}^{\rho\sigma} R^{(\alpha\beta)}{}_{\lambda\mu\nu}$ .

If the indices of these curvature tensors are, however, contracted with other tensors, such as  $F_{\mu M}{}^N$  (3.37) below, which also play the role of projector, it is not necessary to perform the above-mentioned redefinition. This is indeed the case for the calculation of the action integral in the next subsection.

### 3.2 Action Integral for pure spin-3 Gravity

As was shown in [29], the second-order action for spin-3 gravity obtained by substituting (2.3) into the Chern-Simons action is given by

$$\begin{aligned} S_{\text{second-order}} &= \frac{k}{4\pi\ell} \int \text{tr } e \wedge (d\omega(e) + \omega(e) \wedge \omega(e) + \frac{1}{3\ell^2} e \wedge e), \\ &= \frac{k}{48\pi\ell} \int d^3x \left\{ -\epsilon^{\mu\nu\lambda} F_{\mu M}{}^N R^M{}_{N\nu\lambda} + \frac{24}{\ell^2} \tilde{e} \right\} \end{aligned} \quad (3.36)$$

Here  $k = \ell/4G$  and  $G$  is the 3D gravitational constant, and  $\ell$  is the cosmological length related to the cosmological constant by  $\Lambda = -2/\ell^2$ .<sup>9</sup>  $\tilde{e}$  is the cosmological term presented in (3.20). The quantity  $F_{\mu M}^N$  multiplying the generalized curvature is given by

$$F_{\mu M}^N \equiv f_{bc}^a e_{\mu}^c e_M^b E_a^N. \quad (3.37)$$

Because  $e_{(\nu\lambda)}^b$  and  $E_a^{(\nu\lambda)}$  are traceless when contracted with  $g^{\nu\lambda}$  and  $g_{\nu\lambda}$ , respectively, this quantity acts as a projector. So, only the tensor part of the generalized curvature,  $R^M_{N\nu\lambda}$ , contributes to the action integral, and the action integral is invariant under diffeomorphism. Therefore the projection in terms of  $P_{\lambda\rho}^{\mu\nu}$  mentioned in the previous subsection is *not* necessary in this case. To express the action in terms of  $g_{\mu\nu}$  and  $\phi_{\mu\nu\lambda}$ , perturbative expansions for the tensor  $F_{\mu M}^N$  must be worked out. Some of the components are displayed in appendix D.

The resulting second-order action up to  $\mathcal{O}(\phi^3)$  is given by

$$S_{\text{second-order}} = \frac{k}{48\pi} \int d^3x \sqrt{-g} (L_0 + L_2 + \mathcal{O}(\phi^4)). \quad (3.38)$$

Here  $L_0 = 12(\hat{R} + \frac{2}{\ell^2})$  is the Lagrangian for Einstein-Hilbert action. An interesting observation is that not only  $R^\lambda_{\rho\mu\nu}$  but also  $R^{(\lambda\rho)}_{(\kappa\sigma)\mu\nu}$  contribute to  $L_0$ . This modifies the coefficient in front of Einstein-Hilbert action. The next term is given by

$$\begin{aligned} L_2 = & -\frac{8}{\ell^2} \phi_{\mu\nu\rho} \phi^{\mu\nu\rho} + \frac{12}{\ell^2} \phi_{\mu} \phi^{\mu} + 16 \phi_{\mu\lambda\rho} \phi_{\nu}^{\lambda\rho} \hat{R}^{\mu\nu} - 2 \phi_{\mu} \phi_{\nu} \hat{R}^{\mu\nu} - 8 \phi_{\mu\nu\lambda} \phi^{\lambda} \hat{R}^{\mu\nu} \\ & - 4 \phi_{\mu\nu\lambda} \phi^{\mu\nu\lambda} \hat{R} + 2 \phi_{\mu} \phi^{\mu} \hat{R} - 2 \phi_{\mu} \hat{\nabla}^{\mu} \hat{\nabla}_{\nu} \phi^{\nu} + 24 \phi^{\mu\nu\lambda} \hat{\nabla}_{\nu} \hat{\nabla}_{\lambda} \phi_{\mu} \\ & - 16 \phi^{\mu\nu\lambda} \hat{\nabla}_{\lambda} \hat{\nabla}_{\sigma} \phi_{\mu\nu}^{\sigma} + 4 \hat{\nabla}_{\mu} \phi_{\nu} \hat{\nabla}^{\nu} \phi^{\mu} + 4 \hat{\nabla}_{\mu} \phi_{\nu} \hat{\nabla}^{\mu} \phi^{\nu} - 16 \hat{\nabla}_{\mu} \phi^{\mu\nu\lambda} \hat{\nabla}_{\rho} \phi^{\rho}_{\nu\lambda} \\ & + 16 \hat{\nabla}^{\mu} \phi^{\nu} \hat{\nabla}_{\lambda} \phi_{\mu\nu}^{\lambda} - 2 (\hat{\nabla}_{\mu} \phi^{\mu})^2 + 2 \phi^{\mu} \hat{\nabla}_{\nu} \hat{\nabla}_{\mu} \phi^{\nu} - 8 \phi^{\mu\nu\lambda} \hat{\nabla}_{\rho} \hat{\nabla}_{\lambda} \phi_{\mu\nu}^{\rho} \\ & + 8 \phi^{\mu\nu\lambda} \hat{\nabla}^{\rho} \hat{\nabla}_{\rho} \phi_{\mu\nu\lambda} - 4 \hat{\nabla}_{\mu} \phi_{\nu\lambda\rho} \hat{\nabla}^{\rho} \phi^{\nu\lambda\mu} + \frac{20}{3} \hat{\nabla}_{\mu} \phi_{\nu\lambda\rho} \hat{\nabla}^{\mu} \phi^{\nu\lambda\rho}. \end{aligned} \quad (3.39)$$

This Lagrangian can be rewritten into the Fronsdal form by partial integration.

$$\begin{aligned} L_2 = & \frac{4}{3} \left[ \phi^{\mu\nu\rho} (\mathcal{F}_{\mu\nu\rho} - \frac{3}{2} g_{\mu\nu} \mathcal{F}_{\rho}) - \frac{3}{2} \hat{R} \phi_{\mu\nu\rho} \phi^{\mu\nu\rho} + \frac{9}{2} \hat{R}_{\rho\sigma} \phi^{\rho}_{\mu\nu} \phi^{\sigma\mu\nu} - \frac{9}{4} \hat{R}_{\rho\sigma} \phi^{\rho} \phi^{\sigma} \right. \\ & \left. - \frac{6}{\ell^2} \phi_{\mu\nu\rho} \phi^{\mu\nu\rho} + \frac{9}{\ell^2} \phi_{\mu} \phi^{\mu} \right] + \frac{1}{\sqrt{-g}} \partial_{\rho} [\sqrt{-g} Q^{\rho}] \end{aligned} \quad (3.40)$$

$\mathcal{F}_{\mu\nu\rho}$  is the Fronsdal tensor

$$\mathcal{F}_{\mu\nu\rho} = \hat{\nabla}^{\lambda} \hat{\nabla}_{\lambda} \phi_{\mu\nu\rho} - \frac{3}{2} \hat{\nabla}^{\lambda} \hat{\nabla}_{(\mu} \phi_{\nu\rho)\lambda} - \frac{3}{2} \hat{\nabla}_{(\mu} \hat{\nabla}^{\lambda} \phi_{\nu\rho)\lambda} + 3 \hat{\nabla}_{(\mu} \hat{\nabla}_{\nu} \phi_{\rho)}, \quad (3.41)$$

and  $\mathcal{F}_{\mu} = \mathcal{F}_{\mu\nu\lambda} g^{\nu\lambda}$ .  $Q^{\rho}$  in the surface term is given by

$$\begin{aligned} Q^{\rho} = & \frac{20}{3} \phi_{\mu\nu\lambda} \hat{\nabla}^{\rho} \phi^{\mu\nu\lambda} + 4 \phi_{\mu} \hat{\nabla}^{\rho} \phi^{\mu} + 4 \phi_{\mu} \hat{\nabla}^{\mu} \phi^{\rho} - 2 \phi^{\rho} \hat{\nabla}_{\mu} \phi^{\mu} - 4 \phi_{\mu\nu\lambda} \hat{\nabla}^{\lambda} \phi^{\mu\nu\rho} \\ & - 16 \phi^{\mu\nu\rho} \hat{\nabla}^{\lambda} \phi_{\mu\nu\lambda} + 16 \phi^{\mu\nu\rho} \hat{\nabla}_{\nu} \phi_{\mu} + 4 \phi^{\mu\nu\rho} \hat{\nabla}_{\nu} \phi_{\mu} - 4 \phi^{\mu} \hat{\nabla}_{\nu} \phi_{\mu}^{\nu\rho}. \end{aligned} \quad (3.42)$$

<sup>9</sup>The normalization of the action (4.30) in [29] is not appropriate. It must be corrected by a factor 1/4.

In this way the second-order action can be divided into three integrals:

$$S_{\text{second order}} = S_{\text{EH}} + S_{\text{free Fronsdal}} + S_{\text{boundary}} \quad (3.43)$$

The bulk part of (3.40) is the linearized spin-3 Fronsdal action with  $\phi$  mass terms, and agrees with the result of [28].<sup>10</sup>  $S_{\text{boundary}} = \int d^2x \sqrt{-g} Q^r$  connects the CS action in the metric-like formalism and the Einstein-Fronsdal type action in the metric-like formalism.

### 3.3 Transformations of the Metric-Like Quantities

The transformation properties of  $g_{\mu\nu}$  and  $\phi_{\mu\nu\rho}$  were studied in [29]. It was shown that  $g_{\mu\nu}$  transforms as

$$\delta g_{\mu\nu} = \hat{\nabla}_\mu \xi_\nu + \hat{\nabla}_\nu \xi_\mu - \Gamma_{\mu\nu}^{(\lambda\rho)} \zeta_{(\lambda\rho)} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \quad (3.44)$$

$\xi_\mu$  and  $\zeta_{\mu\nu} = \xi_{(\mu\nu)} - G_{(\mu\nu)\rho} g^{\rho\sigma} \xi_\sigma$  are parameters of diffeomorphism and spin-3 transformation.<sup>11</sup> So,  $g_{\mu\nu}$  behaves as a tensor under diffeomorphism. By using (3.28), the transformation rule under spin-3 transformation is also derived.

$$\begin{aligned} \delta_\zeta g_{\mu\nu} = & -\frac{8}{3} \zeta^{\lambda\rho} \hat{\nabla}_{(\mu} \phi_{\nu)\lambda\rho} - \frac{8}{3} \zeta_{(\mu}{}^\lambda \hat{\nabla}_{\nu)} \phi_\lambda - \frac{8}{3} \zeta_{(\mu}{}^\lambda \hat{\nabla}_{|\lambda|} \phi_{\nu)} \\ & + \frac{4}{3} \zeta^{\lambda\rho} \hat{\nabla}_\lambda \phi_{\mu\nu\rho} + \frac{8}{3} \zeta_{(\mu}{}^\lambda \hat{\nabla}^\rho \phi_{\nu)\lambda\rho} + \frac{8}{3} g_{\mu\nu} \zeta^{\lambda\rho} \hat{\nabla}_\lambda \phi_\rho \\ & + \frac{2}{3} \zeta_{\mu\nu} \hat{\nabla}_\lambda \phi^\lambda - \frac{4}{3} g_{\mu\nu} \zeta^{\lambda\rho} \hat{\nabla}^\sigma \phi_{\lambda\rho\sigma} + \mathcal{O}(\phi^3) \end{aligned} \quad (3.45)$$

This result agrees with that of [28].

The case of  $\phi_{\mu\nu\rho} = \frac{1}{4} \text{tr } e_\mu \{e_\nu, e_\rho\}$  is more complicated. Under diffeomorphism it was shown [29] that  $\phi$  transforms as<sup>12</sup>

$$\begin{aligned} \delta_\xi \phi_{\mu\nu\lambda} = & \xi^\sigma \hat{\nabla}_\sigma \phi_{\mu\nu\lambda} + \hat{\nabla}_\mu \xi^\sigma \phi_{\sigma\nu\lambda} + \hat{\nabla}_\nu \xi^\sigma \phi_{\sigma\mu\lambda} + \hat{\nabla}_\lambda \xi^\sigma \phi_{\sigma\mu\nu} + \hat{\nabla}_\mu \xi^\sigma \phi_{\sigma\lambda\nu} \\ & + \frac{1}{5} g_{\mu\nu} \xi^\sigma \left( \hat{\nabla}_\lambda \phi_{\sigma\kappa}{}^\kappa - \hat{\nabla}_\sigma \phi_{\lambda\kappa}{}^\kappa \right) + \frac{1}{5} g_{\nu\lambda} \xi^\sigma \left( \hat{\nabla}_\mu \phi_{\sigma\kappa}{}^\kappa - \hat{\nabla}_\sigma \phi_{\mu\kappa}{}^\kappa \right) \\ & + \frac{1}{5} g_{\mu\lambda} \xi^\sigma \left( \hat{\nabla}_\nu \phi_{\sigma\kappa}{}^\kappa - \hat{\nabla}_\sigma \phi_{\nu\kappa}{}^\kappa \right) \\ & + \left\{ \frac{1}{5} g^{\alpha\beta} (S_{\mu\alpha,\nu\beta} \xi_\lambda + S_{\mu\alpha,\lambda\beta} \xi_\nu) - \xi^\alpha S_{\mu\nu,\lambda\alpha} + \text{cyclic permutations of } \mu, \nu, \lambda \right\}. \end{aligned} \quad (3.46)$$

The appearance of this expression is quite different from the transformation rule of a rank 3 tensor. Thus the transformation rule of  $\phi_{\mu\nu\rho} = \frac{1}{4} \text{tr } e_\mu \{e_\nu, e_\rho\}$  is a nontrivial issue, although

<sup>10</sup>Our  $\phi$  and the spin-3 field  $\varphi$  in [28] are related as  $\phi_{\mu\nu\rho} = 3\varphi_{\mu\nu\rho}$ .

<sup>11</sup> $\xi_M = \frac{1}{2} \text{tr } (\Lambda_- e_M)$ . See eqs (5.4) and (5.5) of [29].  $\zeta_{\mu\nu}$  satisfies  $\zeta_{\mu\nu} g^{\mu\nu} = 0$ .

<sup>12</sup>There is a typo in eq (5.13) of [29]. The coefficient 1/5 at the top of the last line of (3.46) is missing.

the tensorial property of the counterpart,  $g_{\mu\nu} = \frac{1}{2} \text{tr } e_\mu e_\nu$ , is easy to justify. We have checked that  $\phi_{\mu\nu\rho}$  behaves as a tensor up to  $\mathcal{O}(\phi^4)$ :

$$\delta_\xi \phi_{\mu\nu\rho} = \xi_\lambda \hat{\nabla}^\lambda \phi_{\mu\nu\rho} + 3 \phi_{(\mu\nu}{}^\lambda \hat{\nabla}_{\rho)} \xi_\lambda + \mathcal{O}(\phi^5). \quad (3.47)$$

All order proof is not yet obtained.

Finally, we computed the spin-3 transformation of  $\phi$  by using eqs (5.15) and (5.3) of [29].

$$\begin{aligned} \delta_\zeta \phi_{\mu\nu\lambda} &= \partial_\mu \zeta_{\nu\lambda} - \frac{1}{2} \Gamma_{\mu(\nu\lambda)}^{(\rho\sigma)} \zeta_{\rho\sigma} + (\text{cyclic permutations of } \mu, \nu, \lambda) \\ &= 3 \hat{\nabla}_{(\mu} \zeta_{\nu\lambda)} + \mathcal{O}(\phi^2) \end{aligned} \quad (3.48)$$

The 0-th order term of this transformation was first obtained in [28]. We also computed the next  $\mathcal{O}(\phi^2)$  terms. These are non-vanishing. However, they are complicated to display here. The action integral (3.38) is invariant under the above  $\mathcal{O}(\phi^0)$  transformations. This was first shown by [28].

## 4 Matter Coupled to Spin-3 Gravity

In this section we will couple a  $(B, C)$  system composed of 0-form  $C$  and 2-form  $B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$  to the spin-3 gravity topologically. These are  $3 \times 3$  matrices and can be expanded into the basis  $\{t_0, t_a\}$  ( $t_0 = \mathbf{1}$  is an identity matrix.)

$$C = C^A t_A = C^0 t_0 + C^a t_a, \quad (4.1)$$

$$B = B^A t_A = B^0 t_0 + B^a t_a \quad (4.2)$$

The action integral is given by

$$S_{\text{matter}} = \int \text{tr } B \wedge (dC + AC - C\bar{A}). \quad (4.3)$$

This action is invariant under  $SL(3, R) \times SL(3, R)$  gauge transformation:

$$\begin{aligned} A &\rightarrow A' = U^{-1} dU + U^{-1} A U, & \bar{A} &\rightarrow \bar{A}' = \bar{U}^{-1} d\bar{U} + \bar{U}^{-1} \bar{A} \bar{U}, \\ C &\rightarrow C' = U^{-1} C \bar{U}, & B &\rightarrow B' = \bar{U}^{-1} B U. \end{aligned} \quad (4.4)$$

Here  $U$  and  $\bar{U}$  are  $3 \times 3$  matrices corresponding to the first and second  $SL(3, R)$ , respectively. As we will see in (4.14)-(4.15), the 0-th components  $B^0$  and  $C^0$  in (4.1)-(4.2), proportional to  $\mathbf{1}$ , are necessary. By using the vielbein and the spin connection<sup>13</sup>, action integral (4.3) in terms of the components reads

$$\begin{aligned} S_{\text{matter}} &= \int d^3x \epsilon^{\mu\nu\lambda} \text{tr } \frac{1}{2} B_{\mu\nu} (\partial_\lambda C + A_\lambda C - C \bar{A}_\lambda) \\ &= \int d^3x \epsilon^{\mu\nu\lambda} B_{A\mu\nu} (\partial_\lambda C^A + f^A{}_{bc} \omega_\lambda^b C^c + \frac{1}{\ell} d^A{}_{bC} e_\lambda^b C^C) \end{aligned} \quad (4.5)$$

---

<sup>13</sup>  $A = \omega + \ell^{-1} e$  and  $\bar{A} = \omega - \ell^{-1} e$ .

Here  $A, B, C, \dots$  run over  $0, 1, 2, \dots, 8$ , while  $a, b, c, \dots$  over  $1, 2, \dots, 8$ . These indices are raised and lowered by  $h_{AB}$  and its inverse  $h^{AB}$ . The Killing vector  $h_{ab}$  for  $SL(3, R)$  is defined in appendix A of [29] and  $h_{00} = 3/2$  and  $h_{a0} = h_{0a} = 0$ . Hence  $B_0 = \frac{3}{2} B^0$ .  $f^A_{BC}$  vanishes, if at least one of  $A, B, C$  are 0.  $f^a_{bc}$  is the structure constant for  $SL(3, R)$  and  $d^a_{bc}$  the invariant symmetric tensor.  $d^0_{ab} = (4/3) h_{ab}$ ,  $d^a_{b0} = 2 \delta^a_b$ . Let us note that this action does not contain the metric tensor like the Chern-Simons action, hence it is topological: it is invariant under general coordinate transformations in the ordinary sense. This symmetry is independent of the above gauge symmetry.

#### 4.1 Infinitesimal Gauge Transformations

Gauge transformation (4.4) in infinitesimal form will be studied now. Let us write  $U$  and  $\bar{U}$  as  $U = e^\Lambda \approx 1 + \Lambda$ ,  $\bar{U} = e^{\bar{\Lambda}} \approx 1 + \bar{\Lambda}$ . The transformations are classified into two sets; (a) local Lorentz-like transformation  $\Lambda = \bar{\Lambda} = \Lambda_+$ ; (b) local translation (diffeo+spin-3 transformation)  $\Lambda = -\bar{\Lambda} = \Lambda_-$ . Additionally, there is also an extra Abelian gauge symmetry for  $B$ ; (c)  $\delta B = d\Xi$  ( $\Xi = \Xi_\mu dx^\mu$  is a one-form.)

- (a) Under local Lorentz-like transformation,  $e$  and  $\omega$  transform as  $\delta e = [e, \Lambda_+]$  and  $\delta \omega = d\Lambda_+ + [\omega, \Lambda_+]$ . A transformation rule for matter is

$$\delta C = -\Lambda_+ C + C \Lambda_+ = [C, \Lambda_+], \quad (4.6)$$

$$\delta B = B \Lambda_+ - \Lambda_+ B = [B, \Lambda_+]. \quad (4.7)$$

Writing  $\Lambda_+ = t_a \Lambda_+^a$ , the transformation of the components is

$$\delta C^0 = 0, \quad \delta C^a = f^a_{bc} C^b \Lambda_+^c, \quad (4.8)$$

$$\delta B^0 = 0, \quad \delta B^a = f^a_{bc} B^b \Lambda_+^c. \quad (4.9)$$

So the 0-th components of the matter are singlets under local Lorentz-like transformation.

- (b) Under local translation,  $e$  and  $\omega$  transform as

$$\delta e = \ell(d\Lambda_- + [\omega, \Lambda_-]) \equiv \ell D^{(L)} \Lambda_-, \quad (4.10)$$

$$\delta \omega = (1/\ell) [e, \Lambda_-]. \quad (4.11)$$

The matter fields transform as

$$\delta C = -\Lambda_- C - C \Lambda_- = -\{\Lambda_-, C\}, \quad (4.12)$$

$$\delta B = \{\Lambda_-, B\} \quad (4.13)$$

In the transformation of  $C^a$  the 0-th component inevitably appear, and thus we are forced to introduce them from the beginning;

$$\delta C^0 = -d_{bc}^0 \Lambda_-^b C^c = -\frac{4}{3} \Lambda_-^a C_a, \quad (4.14)$$

$$\delta C^a = -d_{bC}^a \Lambda_-^b C^C = -d_{bc}^a \Lambda_-^b C^c - 2 \Lambda_-^a C^0 \quad (4.15)$$

$$\delta B^0 = d_{bc}^0 B^c = \frac{4}{3} \Lambda_-^a B_a, \quad (4.16)$$

$$\delta B^a = d_{bC}^a \Lambda_-^b B^C = d_{bc}^a \Lambda_-^b B^c + 2 \Lambda_-^a B^0 \quad (4.17)$$

Later, we will show that  $C^0 = (1/3)\text{tr}C$  is a scalar under ordinary diffeomorphism, but transforms nontrivially under spin-3 transformation.

(c) Under the two-form gauge transformation,  $B$  changes as follows.

$$\delta B_{\mu\nu} = \partial_\mu \Xi_\nu - \partial_\nu \Xi_\mu \quad (4.18)$$

Out of the three components of  $\Xi_\mu$ , the one which satisfies  $d\Xi = 0$  is redundant, and only two of the three components of  $B_{\mu\nu}$  can be gauged away; it is possible to set  $B_{rt} = B_{r\phi} = 0$ . Therefore  $B$  is essentially a single ‘scalar’field ( $B_{t\phi}$ ) which will be dual to a scalar operator in the boundary CFT.

## 4.2 Spin-3 Gravity with Torsion

Now we will eliminate the spin connection  $\omega_\mu^a$  to obtain the metric-like theory. First, we tried to use the solution  $\omega_\mu^a(e)$ , (2.3). In this case the transformation rules of  $g_{\mu\nu}$  and  $\phi_{\mu\nu\lambda}$  are the same as in the pure spin-3 gravity theory. It is also found that the transformation rules of  $C$  under diffeomorphism<sup>14</sup> coincide with the usual rules for tensors. However, it was found that the spin-3 transformation rules for  $C$ , (4.57) and (4.58) presented below with  $\Delta\Gamma = 0$ , do not leave the action invariant. At present it is not clear how to modify the action and the transformation rules to recover the symmetry.

For this reason we will solve the equation of motion for the total action with respect to  $\omega_\mu^a$ . When the matter fields are coupled to the pure spin-3 gravity, a torsion is introduced. To see this, let us consider our total action in the Palatini formalism.

$$S_{\text{tot}}[e, \omega, B, C] = \frac{k}{4\pi\ell} \int \text{tr} \left( e \wedge R(\omega) + \frac{1}{3\ell^2} e \wedge e \right) + \int \text{tr} \left( B \wedge (dC + \frac{1}{\ell} \{e, C\} + [\omega, C]) \right) \quad (4.19)$$

Here  $R(\omega) = d\omega + \omega \wedge \omega$ . The first integral is the Chern-Simons action.[32][33][15] The equation of motion obtained by variation with respect to  $\omega$  is no longer a torsion-free condition:

$$\partial_\mu e_\nu^a + f_{bc}^a \omega_\mu^b e_\nu^c - (\mu \leftrightarrow \nu) = -T_{\mu\nu}. \quad (4.20)$$

<sup>14</sup>These are given by (4.55) and (4.56) below with  $\Delta\Gamma$  set to zero.



Here  $T_{\mu\nu} = T_{\mu\nu}^M e_M$  is a torsion tensor defined by

$$T_{\mu\nu} = -\frac{4\pi\ell}{k} [B_{\mu\nu}, C]. \quad (4.21)$$

Eq (4.20) defines a new spin connection  $\tilde{\omega}_\mu^a(e, B, C)$  and a new set of connections  $\tilde{\Gamma}_{\mu N}^M$ . The equation

$$\partial_\mu e_\nu^a + f_{bc}^a \tilde{\omega}_\mu^b e_\nu^c = \tilde{\Gamma}_{\mu\nu}^M e_M \quad (4.22)$$

with  $(\tilde{\Gamma}_{\mu\nu}^M - \tilde{\Gamma}_{\nu\mu}^M) e_M = -T_{\mu\nu}$  can be solved for  $\tilde{\Gamma}_{\mu\nu}^M$ , by the same procedure as in sec.3 of [29] and then  $\tilde{\Gamma}_{\mu(\nu\lambda)}^M$  is determined in terms of them. The result can be written in the form

$$\tilde{\Gamma}_{\mu N}^M = \Gamma_{\mu N}^M + \Delta \Gamma_{\mu N}^M, \quad (4.23)$$

where  $\Gamma_{\mu N}^M$  is the connection for the pure spin-3 gravity. For the component,  $\tilde{\Gamma}_{\mu\nu}^{(\kappa\sigma)}$ , the difference is given by

$$\begin{aligned} \Delta \Gamma_{\mu\nu}^{(\kappa\sigma)} = & -\frac{1}{2} T_{\mu\nu}^{(\kappa\sigma)} + \frac{1}{8} J^{(\kappa\sigma)(\lambda\rho)} [T_{\mu\lambda,(\nu\rho)} + T_{\mu\rho,(\nu\lambda)} - 2g_{\lambda\rho} T_{\mu}{}^\sigma{}_{,(\nu\sigma)} \\ & + g_{\nu\rho} T_{\mu}{}^\sigma{}_{,(\lambda\sigma)} + g_{\nu\lambda} T_{\mu}{}^\sigma{}_{,(\rho\sigma)} + T_{\nu\lambda,(\mu\rho)} + T_{\nu\rho,(\mu\lambda)} - 2g_{\lambda\rho} T_{\nu}{}^\sigma{}_{,(\mu\sigma)} \\ & + g_{\mu\rho} T_{\nu}{}^\sigma{}_{,(\lambda\sigma)} + g_{\mu\lambda} T_{\nu}{}^\sigma{}_{,(\rho\sigma)} + g_{\nu\rho} T_{\lambda}{}^\sigma{}_{,(\mu\sigma)} + g_{\mu\rho} T_{\lambda}{}^\sigma{}_{,(\nu\sigma)} \\ & - 2g_{\mu\nu} T_{\lambda}{}^\sigma{}_{,(\rho\sigma)} + g_{\nu\lambda} T_{\rho}{}^\sigma{}_{,(\mu\sigma)} + g_{\mu\lambda} T_{\rho}{}^\sigma{}_{,(\nu\sigma)} - 2g_{\mu\nu} T_{\rho}{}^\sigma{}_{,(\lambda\sigma)}] + \mathcal{O}(\phi^1) \end{aligned} \quad (4.24)$$

The next order contribution,  $\Delta [\Gamma_{\mu\nu}^{(\tau\eta)}]_1 (= \mathcal{O}(\phi^1))$  is displayed in appendix G. The difference of another connection,  $\Delta \Gamma_{\mu\nu}^{\rho\sigma}$ , is related to  $\Delta \Gamma_{\mu\nu}^{(\rho\sigma)}$  by

$$\Delta \Gamma_{\mu\nu}^M G_{M\lambda} = \Delta \Gamma_{\mu\nu}^\rho G_{\rho\lambda} + \frac{1}{2} \Delta \Gamma_{\mu\nu}^{(\rho\sigma)} G_{(\rho\sigma)\lambda} = -\frac{1}{2} (T_{\mu\nu,\lambda} + T_{\lambda\mu,\nu} + T_{\lambda\nu,\mu}), \quad (4.25)$$

In the above equations, we used a notation.

$$T_{\mu\nu,M} = T_{\mu\nu}^N G_{NM} \quad (4.26)$$

Expression (4.25) satisfies  $\Delta \Gamma_{\mu\nu}^M G_{M\lambda} + \Delta \Gamma_{\mu\lambda}^M G_{M\nu} = 0$  due to the anti-symmetry of  $T_{\mu\nu}$ . In (4.24) the indices of  $T_{\mu\nu,(\lambda\rho)}$  are raised by  $g^{\kappa\sigma}$ . Let us note that  $\Delta \Gamma_{\mu\nu}^M$  is not symmetric under interchange of  $\mu$  and  $\nu$ , due to the torsion. The other two connections  $\tilde{\Gamma}_{\mu(\nu\lambda)}^M$  are obtained by replacing  $\Gamma_{\mu N}^M$  by  $\tilde{\Gamma}_{\mu N}^M$  in eqs (4.12) and (4.13) of [29].

The spin connection  $\tilde{\omega}_\mu^a$  which satisfies (4.22) is now given by

$$\tilde{\omega}_\mu^a(e, B, C) \equiv \frac{1}{12} f^{ab}{}_c E_b^M \tilde{\nabla}_\mu e_M^c. \quad (4.27)$$

Here  $\tilde{\nabla}_\mu$  is the covariant derivative associated with  $\tilde{\Gamma}_{\mu N}^M$ .  $G_{MN}$  is compatible with  $\tilde{\nabla}_\mu$ . The generalized curvature tensor which corresponds to the above new connection is defined by

$$\begin{aligned} \tilde{R}_{N\mu\nu}^M &= \partial_\mu \tilde{\Gamma}_{\nu N}^M - \partial_\nu \tilde{\Gamma}_{\mu N}^M + \tilde{\Gamma}_{\mu K}^M \tilde{\Gamma}_{\nu N}^K - \tilde{\Gamma}_{\nu K}^M \tilde{\Gamma}_{\mu N}^K \\ &= R_{N\mu\nu}^M + \nabla_\mu \Delta \Gamma_{\nu N}^M - \nabla_\nu \Delta \Gamma_{\mu N}^M + \Delta \Gamma_{\mu K}^M \Delta \Gamma_{\nu N}^K - \Delta \Gamma_{\nu K}^M \Delta \Gamma_{\mu N}^K \end{aligned} \quad (4.28)$$

In the second line of the above equation, the covariant derivative  $\nabla_\mu \Delta \Gamma_{\nu K}^M$  is used here for brevity with tacit understanding that the non-existing component  $\Delta \Gamma_{(\mu\nu)N}^M = 0$ . After substitution of (4.27) into (4.19), we obtain the total action.

$$\begin{aligned} \tilde{S}_{\text{tot}} \equiv S_{\text{tot}}[e, \tilde{\omega}, B, C] &= \frac{k}{48\pi\ell} \int d^3x \left( -\epsilon^{\mu\nu\lambda} F_{\mu M}{}^N \tilde{R}^M{}_{N\nu\lambda} + \frac{24}{\ell^2} \tilde{e} \right) \\ &+ \int \text{tr } B \wedge \left( dC + \frac{1}{\ell} \{e, C\} + [\tilde{\omega}, C] \right) \end{aligned} \quad (4.29)$$

When the generalized curvature tensor (4.28) is substituted into the above equation, those terms linear in  $\nabla_\nu \Delta \Gamma_{\lambda N}^M$  turn out total derivative ones, and can be dropped. This is because the metric-like quantity  $F_{\mu M}{}^N$  is made of the vielbeins, and covariantly constant:  $\nabla_\nu F_{\mu M}{}^N = D_\nu F_{\mu M}{}^N = 0$ .<sup>15</sup> Finally, only the quadratic terms remain, and the total action is given by

$$\begin{aligned} \tilde{S}_{\text{tot}} &= S_{\text{second-order}} + \frac{k}{48\pi\ell} \int d^3x \left( -2\epsilon^{\mu\nu\lambda} F_{\mu M}{}^N \Delta \Gamma_{\nu K}^M \Delta \Gamma_{\lambda N}^K \right) \\ &+ \int \text{tr } B \wedge \left( dC + \frac{1}{\ell} \{e, C\} + [\tilde{\omega}, C] \right) \end{aligned} \quad (4.30)$$

The first term in the first line is the pure spin-3 gravity action (3.36). The last term in the first line is quadratic in the torsion tensor and depends on  $B$  and  $C$ .

### 4.3 Proof of Local Translation Invariance of the Action

In order to prove that  $\tilde{S}_{\text{tot}} = S_{\text{tot}}[e, \tilde{\omega}, B, C]$  is invariant under the local translation (4.10), (4.12)-(4.13), we will use the 1.5 order formalism. Under variation of the fields,  $\tilde{S}_{\text{tot}}$  changes as follows.

$$\delta \tilde{S}_{\text{tot}} = \frac{\delta S_{\text{tot}}}{\delta e} \delta e + \frac{\delta S_{\text{tot}}}{\delta B} \delta B + \frac{\delta S_{\text{tot}}}{\delta C} \delta C + \frac{\delta S_{\text{tot}}}{\delta \omega} \delta \tilde{\omega}[e, B, C] \quad (4.31)$$

After the functional differentiations, we set  $\omega = \tilde{\omega}$  on the righthand side. The last variation  $\delta \tilde{\omega}[e, B, C]$  is computed according to the dependence of the solution  $\omega = \tilde{\omega}$  on the other fields,  $e_\mu$ ,  $B$  and  $C$ . This is actually very complicated, but because  $\tilde{\omega}$  solves the equation of motion, the functional derivative multiplying this variation vanishes at  $\omega = \tilde{\omega}$ . This means that when calculating the variation, we can keep  $\tilde{\omega}$  fixed.

For the pure spin-3 gravity part of the action, we only need to vary the vielbein. The  $e \wedge R(\tilde{\omega})$  part is invariant up to total derivative terms, because of (4.10) and the Bianchi identity,  $dR(\omega) + \omega \wedge R(\omega) - R(\omega) \wedge \omega = 0$ . The variation of the generalized cosmological term (3.20) is a total derivative:  $\delta \tilde{e} = \partial_\mu (\frac{1}{2} \epsilon^{\mu\nu\lambda} f_{abc} \Lambda_-^a e_\nu^b e_\lambda^c)$ .

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<sup>15</sup>  $D_\nu$  is the full covariant derivative.

For the matter part, if the spin connection is kept fixed, we obtain after simple calculation, the variation of the matter Lagrangian,

$$\epsilon^{\mu\nu\lambda} f_{abc} e_\mu^b \Lambda_-^c [B_{\nu\lambda}, C]^a = \frac{k}{4\pi\ell} \epsilon^{\mu\nu\lambda} f_{abc} e_\mu^b \Lambda_-^c T_{\nu\lambda}^a. \quad (4.32)$$

Here the definition of the torsion (4.21) is used. Due to (4.20), this is a total derivative like the variation of the generalized cosmological term mentioned above. Therefore the invariance of  $S_{\text{tot}}[e, \tilde{\omega}, B, C]$  is proved.

#### 4.4 World-volume Components of the Matter

The matter fields,  $C^a$  and  $B^a$ , introduced above have an internal index  $a$  and transform non-trivially under local Lorentz-like transformation (a). By contracting these fields with the generalized vielbein  $e_\mu^a$  and  $e_{(\mu\nu)}^a$ , we will obtain fields which are neutral to local Lorentz-like transformation. The new fields are  $C^0$ ,  $C^M = E_a^M C^a$  and  $B^0$ ,  $B_M = e_M^a B^a$ .  $C^a$  and  $C^M$  are in one-to-one correspondence with each other:  $C^a = C^M e_M^a$ .

We now introduce a Lorentz covariant derivative for  $C^a$ .

$$\begin{aligned} \tilde{D}_\mu^{(L)} C^a &= \partial_\mu C^a + f_{bc}^a \tilde{\omega}_\mu^b C^c, \\ \tilde{D}_\mu^{(L)} C^0 &= \partial_\mu C^0. \end{aligned} \quad (4.33)$$

Similar definition is made for  $\tilde{D}_\mu^{(L)} B_{\nu\lambda}^a$  and  $\tilde{D}_\mu^{(L)} B_{\nu\lambda}^0$ . By replacing  $\tilde{D}_\mu^{(L)}$  on  $C^a$  by the full covariant derivative  $\tilde{D}_\mu$  and using  $C^a = C^M e_M^a$ , recalling that  $e_M^a$  is covariantly constant, we obtain

$$\tilde{D}_\lambda^{(L)} C^a = \tilde{D}_\lambda (C^M e_M^a) = e_M^a \tilde{\nabla}_\lambda C^M. \quad (4.34)$$

Here  $\tilde{\nabla}_\mu$  is the covariant derivative corresponding to  $\tilde{\Gamma}_{\mu N}^M$ .

By assembling the results of the above replacements, matter action (4.5) takes the form.

$$\begin{aligned} S_{\text{matter}} &= \int d^3x \epsilon^{\mu\nu\lambda} \left[ B_{N\mu\nu} E_a^N \left( e_M^a \tilde{\nabla}_\lambda C^M + \frac{1}{\ell} d_{bc}^a e_\lambda^b C^c + \frac{2}{\ell} e_\lambda^a C^0 \right) \right. \\ &\quad \left. + B_{0\mu\nu} \left( \partial_\lambda C^0 + \frac{4}{3\ell} e_\lambda^b C_b \right) \right]. \end{aligned} \quad (4.35)$$

We now rewrite  $B$  as

$$B_N{}^\lambda \equiv \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\lambda} B_{N\mu\nu} = \varepsilon^{\mu\nu\lambda} B_{N\mu\nu}, \quad (4.36)$$

$$B_0{}^\lambda \equiv \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\lambda} B_{0\mu\nu} = \varepsilon^{\mu\nu\lambda} B_{0(\mu\nu)}. \quad (4.37)$$

Here  $\varepsilon^{\mu\nu\lambda}$  is a completely anti-symmetric tensor, and  $g = \det g_{\mu\nu}$  is the determinant of the

metric tensor. We have

$$\begin{aligned}
S_{\text{matter}} &= \int d^3x \sqrt{-g} \left[ B_N^\lambda \left( \tilde{\nabla}_\lambda C^N + \frac{1}{\ell} K^N_{M\lambda} C^M \right) + \frac{2}{\ell} B_\lambda^\lambda C^0 \right. \\
&\quad \left. + B_0^\lambda \left( \partial_\lambda C^0 + \frac{4}{3\ell} C_\lambda \right) \right] \\
&= \int d^3x \sqrt{-g} \left[ B_N^\lambda \left( \nabla_\lambda C^N + \frac{1}{\ell} K^N_{M\lambda} C^M \right) + \frac{2}{\ell} B_\lambda^\lambda C^0 \right. \\
&\quad \left. + B_0^\lambda \left( \partial_\lambda C^0 + \frac{4}{3\ell} C_\lambda \right) + B_N^\lambda C^M \Delta \Gamma_{\lambda M}^N \right] \tag{4.38}
\end{aligned}$$

Here the indices  $M, N$  of  $C$  are raised and lowered in terms of  $G_{MN}$  and  $G^{MN}$ :  $C_\lambda = G_{\lambda N} C^N$ .<sup>16</sup> After the second equality,  $\tilde{\nabla}_\lambda$  is replaced by  $\nabla_\lambda$  and a new term including  $\Delta \Gamma_{\lambda M}^N$  appeared due to the difference of the connections. This is a fourth order interaction  $(BC)^2$  of the matter fields. Also  $K^N_{ML}$  is a metric-like quantity defined by

$$K^N_{ML} \equiv d^a_{bc} E_a^N e_M^b e_L^c \tag{4.39}$$

This quantity has the following expansion in powers of  $\phi$ .

$$K^\mu_{\nu\lambda} = \frac{2}{3} g_{\nu\lambda} \phi^\mu + \mathcal{O}(\phi^3), \tag{4.40}$$

$$K^{(\mu\nu)}_{\lambda\rho} = 2 \left( \delta_\lambda^\mu \delta_\rho^\nu + \delta_\lambda^\nu \delta_\rho^\mu - \frac{2}{3} g^{\mu\nu} g_{\lambda\rho} \right) + \mathcal{O}(\phi^2) \tag{4.41}$$

$K^\mu_{(\nu\lambda)\rho}$  and  $K^{\mu\nu}_{(\lambda\rho)\kappa}$  are already displayed in (3.30) and (3.32). The other components are presented in appendix B. In appendix E, expansion of the part of the matter action, which does not depend on the torsion, is presented in powers of  $\phi$  up to  $\mathcal{O}(\phi^1)$ . Those parts which comes from  $\Delta \Gamma_{\mu M}^N$  turn out too complicated to write down.

To summarize, the total action is given by

$$\begin{aligned}
\tilde{S}_{\text{tot}} &= S_{\text{second-order}} \\
&\quad + \int d^3x \sqrt{-g} \left[ B_N^\lambda \left( \nabla_\lambda C^N + \frac{1}{\ell} K^N_{M\lambda} C^M \right) + \frac{2}{\ell} B_\lambda^\lambda C^0 + B_0^\lambda \left( \partial_\lambda C^0 + \frac{4}{3\ell} C_\lambda \right) \right] \\
&\quad + \int d^3x \left[ \sqrt{-g} B_N^\lambda C^M \Delta \Gamma_{\lambda M}^N - \frac{k}{24\pi\ell} \epsilon^{\mu\nu\lambda} F_{\mu M}{}^N \Delta \Gamma_{\nu K}^M \Delta \Gamma_{\lambda N}^K \right] \tag{4.42}
\end{aligned}$$

Terms in the third line come from the torsion and represent matter interactions.

## 4.5 Symmetry of the Matter-Coupled Theory

The transformation rules for the metric-like fields  $g_{\mu\nu}$  and  $\phi_{\mu\nu\lambda}$  under the local translation (b) are modified from (3.44)-(3.48), because of the torsion terms. The new rules are obtained by replacing  $\nabla_\mu$  and  $\Gamma_{\mu N}^M$  by  $\tilde{\nabla}_\mu$  and  $\tilde{\Gamma}_{\mu N}^M$  in the above equations, respectively. As

<sup>16</sup> Since  $C^0$  is not invariant under spin-3 transformation, a suitable covariant derivative for  $C^0$  needs to be devised like  $\nabla_\mu \rho^a$  defined in eq(4.8) of [29]. This will not be attempted in this paper.

a result, the local translations of  $g_{\mu\nu}$  and  $\phi_{\mu\nu\lambda}$  depend on the matter fields. It might be puzzling that even the diffeomorphism of the metric-like quantities depend on the matter fields through the torsion tensor. For example,

$$\begin{aligned}\delta g_{\mu\nu} &= \tilde{\nabla}_\mu \xi_\nu + \tilde{\nabla}_\nu \xi_\mu \\ &= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \Delta\Gamma_{\mu\nu}^M \xi_M - \Delta\Gamma_{\nu\mu}^M \xi_M.\end{aligned}\quad (4.43)$$

For diffeomorphism,  $\xi^{(\mu\nu)} = 0$  and  $\xi_M = G_{M\lambda} \xi^\lambda$ . This transformation rule is not geometrical one. However, as we metioned before, our theory is originally a topological theory, and has its own diffeomorphism invariance.  $\tilde{S}_{\text{tot}}$  is invariant under diffeomorphism, if  $g_{\mu\nu}$  and  $\phi_{\mu\nu\lambda}$  as well as  $C^0$ ,  $C^\mu$ ,  $C^{(\mu\nu)}$  and  $B_0^\lambda$ ,  $B_\mu^\lambda$  and  $B_{(\mu\nu)}^\lambda$  transform as tensors in the usual way.

The difference between the diffeomorphism and the diffeomorphism-like transformation (4.43) should also be the symmetry transformation of the action. Therefore the symmetry of the matter-coupled theory is larger than that of the pure spin-3 gravity! To study this symmetry in more details, we need to compute the symmetry algebra. This will be left for future. The fact that the diffeomorphism which derives from the topological nature of the Chern-Simons theory and that from the local translation coincide in the pure spin-3 gravity seems accidental. It is an important fact that in order to obtain higher-spin gravity theory with matters, which is invariant under diffeomorphism, matter action in the frame-like approach must be a topological theory.

Similarly, the spin-3 transformations of  $g_{\mu\nu}$  and  $\phi_{\mu\nu\lambda}$  depend on the matter fields through  $\Delta\Gamma_{\mu N}^M$ . For example, the transformation of the metric tensor is (4.43) with  $\xi_M = \frac{1}{2} G_{M(\lambda\rho)} \xi^{(\lambda\rho)}$ .

We now turn to the transformation rule of  $C^A$  under the local translation (b). Instead of the gauge parameter  $\Lambda_-$ , let us introduce new functions  $\xi_M = \ell \Lambda_-^a e_{aM}$  and  $\xi^M = \ell \Lambda_-^a E_a^M$ . Then (4.14) is rewritten as

$$\delta C^0 = -\frac{4}{3\ell} \xi_M C^M. \quad (4.44)$$

This does not look like a diffeomorphism of matter fields. However, if  $\xi^{(\mu\nu)} = 0$ , the eq of motion (4.49) can be used to show that

$$\begin{aligned}\xi^\mu \partial_\mu C^0 &= (\ell E_a^\mu \Lambda_-^a) \left(-\frac{4}{3\ell} e_{b\mu} C^b\right) = -\frac{4}{3} \Lambda_-^a \left(h_{ab} - \frac{1}{2} E_a^{(\mu\nu)} e_{b(\mu\nu)}\right) C^b \\ &= -\frac{4}{3} \Lambda_-^a C_a + \frac{2}{3} \xi^{(\mu\nu)} e_{b(\mu\nu)} C^b = -\frac{4}{3\ell} \xi^M C_M = \delta C^0\end{aligned}\quad (4.45)$$

At the second equality of the first line, a relation  $E_a^M e_M^b = \delta_a^b$ , which is a counterpart of (2.2), is used. Therefore  $\delta C^0$  is the ordinary diffeomorphism for  $C^0$ . On the other hand, if

$\xi^\mu = 0$ , we have

$$\begin{aligned}\delta C^0 &= -\frac{2}{3\ell} \xi^{(\mu\nu)} C_{(\mu\nu)} = -\frac{2}{3\ell} \xi^{(\mu\nu)} G_{(\mu\nu)M} C^M \\ &= -\frac{1}{6\ell} \xi^{(\mu\nu)} C^{(\lambda\rho)} g_{\mu\lambda} g_{\nu\rho} - \frac{2}{3\ell} \xi^{(\mu\nu)} C^\lambda \phi_{\mu\nu\lambda} + \mathcal{O}(\phi^2)\end{aligned}\quad (4.46)$$

$C^0$  mixes with  $C^M$  under spin-3 transformation.

The transformation of  $C^M$  is more involved. Let us recall the transformations of the vielbeins. The vielbein transforms as  $\delta e_\mu^a = \ell \tilde{D}_\mu^{(L)} \Lambda_-^a$ . The transformation of the additional one,  $e_{(\mu\nu)}^a = \frac{1}{4} d^a_{bc} e_\lambda^b e_\rho^c P_{\mu\nu}^{\lambda\rho}$  is given by

$$\delta e_{(\mu\nu)}^a = \frac{1}{4} d^a_{bc} e_\lambda^b e_\rho^c \delta P_{\mu\nu}^{\lambda\rho} + \frac{1}{2} d^a_{bc} E^{bM} e_\rho^c \tilde{\nabla}_\lambda \xi_M P_{\mu\nu}^{\lambda\rho}. \quad (4.47)$$

Then by using (4.15) and (3.44), we have

$$\begin{aligned}\delta C^M = \delta (E_a^M C^a) &= -\frac{1}{\ell} K^M_{NL} C^L \xi^N - C^\mu P^M_N \tilde{\nabla}_\mu \xi^M - \frac{1}{2} C^{(\mu\nu)} K^M_{N\mu} \tilde{\nabla}_\nu \xi^N \\ &\quad + \frac{1}{6} C^{(\mu\nu)} K^M_{\lambda\rho} g^{\lambda\rho} \tilde{\nabla}_\mu \xi_\nu - \frac{2}{\ell} C^0 \xi^M + C^N \delta(P^M_N).\end{aligned}\quad (4.48)$$

Here  $\xi_M$  and  $\xi^N$  are related as  $\xi_M = G_{MN} \xi^N$ . If  $M = (\mu\nu)$  and  $N = (\lambda\rho)$ ,  $P^M_N = P_{\lambda\rho}^{\mu\nu}$ . If  $M = \mu$  and  $N = \nu$ ,  $P^M_N = \delta^\mu_\nu$ . Otherwise,  $P^M_N = 0$ .

Because the matter system is topological, it may be allowed to use the equations of motion to rewrite the above transformation. The equations of motion for  $C^0$  and  $C^a$  derived from (4.5) are as follows.

$$\partial_\mu C^0 + \frac{4}{3\ell} e_{a\mu} C^a = 0, \quad (4.49)$$

$$\tilde{D}_\mu^{(L)} C^a + \frac{1}{\ell} d^a_{bc} e_\mu^b C^c + \frac{2}{\ell} e_\mu^a C^0 = 0 \quad (4.50)$$

Those for  $B$ 's are

$$\partial_{[\mu} B_{\nu\lambda]}^0 - \frac{4}{3\ell} e_{a[\mu} B_{\nu\lambda]}^a = 0, \quad (4.51)$$

$$\tilde{D}_{[\mu}^{(L)} B_{\nu\lambda]}^a - \frac{1}{\ell} d^a_{bc} e_{[\mu}^b B_{\nu\lambda]}^c - \frac{2}{\ell} e_{[\mu}^a B_{\nu\lambda]}^0 = 0. \quad (4.52)$$

Here  $[ , ]$  stands for complete anti-symmetrization of the indices in between.

Eq (4.50) leads to the relations.

$$\tilde{\nabla}_\mu C^\nu = -\frac{1}{\ell} K^\nu_{\mu M} C^M - \frac{2}{\ell} \delta_\mu^\nu C^0, \quad (4.53)$$

$$\tilde{\nabla}_\mu C^{(\nu\lambda)} = -\frac{1}{\ell} K^{(\nu\lambda)}_{\mu M} C^M + \frac{1}{3} g^{\nu\lambda} g_{\rho\sigma} \tilde{\nabla}_\mu C^{(\rho\sigma)}. \quad (4.54)$$

Note that the second term on the righthand side of the last equation does not vanish, because the two indices of  $g_{\rho\sigma}$  is contracted with the single index  $M = (\rho\sigma)$  of  $C^M$ . This is not the ordinary rule of contraction of indices, and so  $g_{\rho\sigma}$  cannot go through  $\tilde{\nabla}_\mu$ .

When  $\xi^{(\mu\nu)} = 0$ , by using (4.53) and (4.54), we obtain the transformations,<sup>17</sup>

$$\begin{aligned}
\delta C^\mu &= \xi^\rho \tilde{\nabla}_\rho C^\mu - C^\rho \tilde{\nabla}_\rho \xi^\mu - \frac{1}{2} C^{(\rho\sigma)} K^\mu{}_{M\rho} \tilde{\nabla}_\sigma \xi^M + \frac{1}{6} C^{(\rho\sigma)} K^\mu{}_{\kappa\tau} g^{\kappa\tau} \tilde{\nabla}_\rho \xi_\sigma, \\
&= \xi^\rho \partial_\rho C^\mu - C^\rho \partial_\rho \xi^\mu + \frac{1}{2} C^{(\rho\sigma)} K^\mu{}_{M\rho} \Delta \Gamma_{\sigma\kappa}^M \xi^\kappa - \frac{1}{6} C^{(\rho\sigma)} K^\mu{}_{\kappa\tau} g^{\kappa\tau} \Delta \Gamma_{\rho\sigma}^M G_{M\lambda} \xi^\lambda \\
&\quad + \xi^\rho \Delta \Gamma_{\rho M}^\mu C^M - C^\rho \Delta \Gamma_{\rho\lambda}^\mu \xi^\lambda + \mathcal{O}(\phi^2), \\
\delta C^{(\mu\nu)} &= \xi^\rho \tilde{\nabla}_\rho C^{(\mu\nu)} - C^\rho \tilde{\nabla}_\rho \xi^{(\mu\nu)} + \frac{1}{3} g^{\mu\nu} g_{\rho\sigma} C^\lambda \tilde{\nabla}_\lambda \xi^{(\rho\sigma)} - \frac{1}{2} C^{(\lambda\rho)} K^{(\mu\nu)}{}_{M\lambda} \tilde{\nabla}_\rho \xi^M \\
&\quad + \frac{1}{6} C^{(\lambda\rho)} K^{(\mu\nu)}{}_{\sigma\kappa} g^{\sigma\kappa} \tilde{\nabla}_\lambda \xi_\rho - \frac{2}{3} g^{\mu\nu} C^{(\lambda\rho)} \tilde{\nabla}_\lambda \xi_\rho - \frac{1}{3} g^{\mu\nu} g_{\kappa\sigma} \xi^\rho \tilde{\nabla}_\rho C^{(\kappa\sigma)} \\
&= \xi^\rho \partial_\rho C^{(\mu\nu)} - C^{(\mu\rho)} \partial_\rho \xi^\nu - C^{(\rho\nu)} \partial_\rho \xi^\mu + \xi^\rho \Delta \Gamma_{\rho M}^{(\mu\nu)} C^M - C^\rho \Delta \Gamma_{\rho\lambda}^{(\mu\nu)} \xi^\lambda \\
&\quad + \frac{1}{3} g^{\mu\nu} g_{\rho\sigma} C^\lambda \Delta \Gamma_{\lambda\kappa}^{(\rho\sigma)} \xi^\kappa - \frac{1}{2} C^{(\lambda\rho)} K^{(\mu\nu)}{}_{M\lambda} \Delta \Gamma_{\rho\sigma}^M \xi^\sigma \\
&\quad - \frac{1}{6} C^{(\lambda\rho)} K^{(\mu\nu)}{}_{\sigma\kappa} g^{\sigma\kappa} \Delta \Gamma_{\lambda\rho}^M G_{M\tau} \xi^\tau + \frac{2}{3} g^{\mu\nu} C^{(\lambda\rho)} \Delta \Gamma_{\lambda\rho}^M G_{M\tau} \xi^\tau \\
&\quad - \frac{1}{3} g^{\mu\nu} g_{\kappa\sigma} \xi^\rho \Delta \Gamma_{\rho M}^{(\kappa\sigma)} C^M + \mathcal{O}(\phi^2).
\end{aligned} \tag{4.55}$$

In both equations, those terms containing  $\phi$  cancelled in a non-trivial way except in the torsion terms containing  $\Delta\Gamma$ 's. Except for those terms with  $\Delta\Gamma$ , these transformations are those for a vector and a rank-2 tensor. This is the same situation as for the diffeomorphism of  $\phi_{\mu\nu\rho}$  discussed at the end of the subsec.3.4. If there is no torsion, the above transformation coincides with diffeomorphism. The torsion terms are cubic in the matter fields. Due to the torsion terms, the local translation of  $C^\mu$  and  $C^{(\mu\nu)}$  with  $\xi^{(\lambda\rho)} = 0$  does not coincide with the diffeomorphism for a vector and a rank-2 tensor. The matter action (E.1), however, is clearly invariant under the diffeomorphism.<sup>18</sup> Therefore  $\tilde{S}_{\text{tot}}$  is also invariant under the usual diffeomorphism, although it is different from the local translation (b). As we discussed at the beginning of this subsection, the symmetry of the spin-3 gravity coupled to matter is larger than that of the pure spin-3 gravity. Let us also note that in the above equation, although  $\xi^{(\mu\nu)} = 0$ , quantities such as  $\tilde{\nabla}_\lambda \xi^{(\rho\sigma)} = \tilde{\Gamma}_{\lambda\kappa}^{(\rho\sigma)} \xi^\kappa$  do not vanish.

On the other hand, when  $\xi^\mu = 0$ , we have the spin-3 transformation.

$$\begin{aligned}
\delta C^\mu &= -\frac{1}{2\ell} K^\mu{}_{(\nu\rho)M} C^M \xi^{(\nu\rho)} - C^\nu \tilde{\nabla}_\nu \xi^\mu - \frac{1}{2} C^{(\lambda\rho)} K^\mu{}_{M\lambda} \tilde{\nabla}_\rho \xi^M \\
&\quad + \frac{1}{6} C^{(\lambda\rho)} K^\mu{}_{\nu\sigma} g^{\nu\sigma} \tilde{\nabla}_\lambda \xi_\rho \\
&= [\delta C^\mu]_0 + [\delta C^\mu]_1 - \frac{1}{2} C^\nu \Delta \Gamma_{\nu(\lambda\sigma)}^\mu \xi^{(\lambda\sigma)} - \frac{1}{4} C^{(\lambda\rho)} K^\mu{}_{M\lambda} \Delta \Gamma_{\rho(\sigma\kappa)}^M \xi^{(\sigma\kappa)} \\
&\quad - \frac{1}{6} C^{(\lambda\rho)} K^\mu{}_{\nu\sigma} g^{\nu\sigma} \Delta \Gamma_{\lambda\rho}^M \xi_M + \mathcal{O}(\phi^2),
\end{aligned} \tag{4.57}$$

<sup>17</sup>Relation with  $\xi_\mu$ ,  $\zeta_{\mu\nu}$  in subsec.3.3 is,  $\xi_\mu = g_{\mu\nu} \xi^\nu$  in the case of diffeomorphism ( $\xi^{(\mu\nu)} = 0$ ), and  $\zeta_{\mu\nu} = \frac{1}{2} M_{(\mu\nu)(\lambda\rho)} \xi^{(\lambda\rho)}$  in the case of spin-3 transformation ( $\xi^\mu = 0$ ).

<sup>18</sup>  $T_{\mu\nu}^M$  and  $\Delta\Gamma_{\mu M}^N$  behave as tensors under diffeomorphism.

$$\begin{aligned}
\delta C^{(\mu\nu)} &= -\frac{1}{2\ell} K^{(\mu\nu)}_{(\lambda\rho)M} C^M \xi^{(\lambda\rho)} - \frac{2}{\ell} C^0 \xi^{(\mu\nu)} - C^\rho \tilde{\nabla}_\rho \xi^{(\mu\nu)} \\
&\quad - \frac{1}{2} C^{(\lambda\rho)} K^{(\mu\nu)}_{M\lambda} \tilde{\nabla}_\rho \xi^M + \frac{1}{6} C^{(\lambda\rho)} K^{(\mu\nu)}_{\sigma\kappa} g^{\sigma\kappa} \tilde{\nabla}_\lambda \xi_\rho \\
&\quad - \frac{2}{3} g^{\mu\nu} C^{(\lambda\rho)} \tilde{\nabla}_\rho \xi_\lambda + \frac{1}{3} g^{\mu\nu} g_{\sigma\kappa} C^\lambda \tilde{\nabla}_\lambda \xi^{(\sigma\kappa)} \\
&= \left[ \delta C^{(\mu\nu)} \right]_0 + \left[ \delta C^{(\mu\nu)} \right]_1 - \frac{1}{2} C^\rho \Delta \Gamma_{\rho(\sigma\kappa)}^{(\mu\nu)} \xi^{(\sigma\kappa)} \\
&\quad - \frac{1}{4} C^{(\lambda\rho)} K^{(\mu\nu)}_{M\lambda} \Delta \Gamma_{\rho(\sigma\kappa)}^M \xi^{(\sigma\kappa)} - \frac{1}{6} C^{(\lambda\rho)} K^{(\mu\nu)}_{\sigma\kappa} g^{\sigma\kappa} \Delta \Gamma_{\lambda\rho}^M \xi_M \\
&\quad + \frac{2}{3} g^{\mu\nu} C^{(\lambda\rho)} \Delta \Gamma_{\rho\lambda}^M \xi_M + \frac{1}{6} g^{\mu\nu} g_{\sigma\kappa} C^\lambda \Delta \Gamma_{\lambda(\tau\eta)}^{(\sigma\kappa)} \xi^{(\tau\eta)} + \mathcal{O}(\phi^2). \quad (4.58)
\end{aligned}$$

$[\delta C^M]_n$  is of  $\mathcal{O}(\phi^n)$ , and does not depend on  $\Delta \Gamma_{\mu N}^M$ . The  $\phi$  expansions of these terms are presented in appendix F. The explicit forms of the terms coming from the torsion is not worked out here explicitly due mainly to the page size.

Transformation of the field  $B$  can also be worked out. As for diffeomorphism, because the matter action (E.1) is manifestly invariant,  $B^{0\mu}$  and  $B_M^\mu = B_{\nu}^\mu$ ,  $B_{(\nu\lambda)}^\mu$  must also transform as tensors. As for the spin-3 transformation, analysis similar to the  $C$  fields lead to the following transformations (with  $\xi^\mu = 0$ ).

$$\delta B_0^\lambda = \frac{1}{\ell} \xi^{(\mu\nu)} B_{(\mu\nu)}^\lambda - \frac{1}{2} g^{\rho\tau} G_{\tau(\kappa\sigma)} \tilde{\nabla}_\rho \xi^{(\kappa\sigma)} B_0^\lambda, \quad (4.59)$$

$$\begin{aligned}
\delta B_\rho^\lambda &= -\frac{1}{2} g^{\alpha\beta} G_{\beta(\kappa\sigma)} \tilde{\nabla}_\alpha \xi^{(\kappa\sigma)} B_\rho^\lambda + \frac{1}{2} B_\tau^\lambda \tilde{\Gamma}_{\rho(\kappa\sigma)}^\tau \xi^{(\kappa\sigma)} + \frac{1}{2} B_{(\kappa\sigma)}^\lambda \tilde{\nabla}_\rho \xi^{(\kappa\sigma)} \\
&\quad + \frac{1}{2\ell} \xi^{(\kappa\sigma)} K_{(\kappa\sigma)\rho}^\tau B_\tau^\lambda + \frac{1}{4\ell} \xi^{(\kappa\sigma)} K_{(\kappa\sigma)\rho}^{(\alpha\beta)} B_{(\alpha\beta)}^\lambda \\
&\quad + \frac{2}{3\ell} G_{\rho(\kappa\sigma)} \xi^{(\kappa\sigma)} B_0^\lambda, \quad (4.60)
\end{aligned}$$

$$\begin{aligned}
\delta B_{(\rho\sigma)}^\lambda &= -\frac{1}{2} g^{\alpha\beta} G_{\beta(\kappa\eta)} \tilde{\nabla}_\alpha \xi^{(\kappa\eta)} B_{(\rho\sigma)}^\lambda - \frac{1}{6} K^M_{\kappa\eta} B_M^\lambda g^{\kappa\eta} (G_{\sigma N} \tilde{\nabla}_\rho \xi^N + G_{\rho N} \tilde{\nabla}_\sigma \xi^N) \\
&\quad + \frac{1}{3} K^M_{\kappa\eta} B_M^\lambda g_{\rho\sigma} g^{\kappa\alpha} g^{\eta\beta} G_{\beta N} \tilde{\nabla}_\alpha \xi^N + \frac{1}{2} K^N_{M\kappa} \tilde{\nabla}_\eta \xi^M P_{\rho\sigma}^{\kappa\eta} B_N^\lambda \\
&\quad + \frac{1}{\ell} K_{(\rho\sigma)M}^N \xi^M B_N^\lambda + \frac{2}{3\ell} G_{(\rho\sigma)(\kappa\tau)} \xi^{(\kappa\tau)} B_0^\lambda \quad (4.61)
\end{aligned}$$

In AdS/CFT correspondence[38][39][40],  $C^0$  will serve as a source for a scalar operator  $O$  on the boundary.  $B_{0t\phi}$  will be the source for its conjugate,  $O^\dagger$ . As in the case of AdS/CFT duality for spinors, we can choose  $\int_{\partial M} \text{Tr}(B \wedge C)$  as a boundary action.[37]<sup>19</sup> What is the role of the other components of  $C$  in the gravity/CFT correspondence? It is known that in spin-3 gravity, there exists  $W_3$  current in the CFT on the boundary[15][34][?] and in the CFT with  $W$  symmetry, the OPEs of the primary field  $O$  and the  $W_3$  current contain new fields,  $\hat{W}_{-1} O$  and  $\hat{W}_{-2} O$ , which are not simply related to  $O$  by just differentiations. [31]

$$W(z) O(z') = \frac{w}{(z-z')^3} O(z') + \frac{1}{(z-z')^2} \hat{W}_{-1} O(z') + \frac{1}{z-z'} \hat{W}_{-2} O(z') + \dots \quad (4.62)$$

<sup>19</sup> $\partial M$  is the boundary of the spacetime.



Naturally,  $C^\mu$  and  $C^{(\mu\nu)}$  are expected to be the sources for  $\hat{W}_{-1} O$  and  $\hat{W}_{-2} O$ , and their anti-chiral counterparts, where  $W_n$  is the expansion mode of the  $W_3$  current in the boundary CFT. Question is whether it is possible to set  $\delta$ -function-type boundary conditions on all these fields at the same time in a single set of solutions. Furthermore, these operators are  $W$  descendants, not primary. So if the correspondence is true, there may be difference from the conventional duality.

In the AdS/CFT correspondence, the solutions to the equations of motion are substituted into the action integral. As in the case of pure spin-3 gravity, the equations of motion in the frame-like approach will be easier to solve than those in the metric-like approach. The equations of motion for the gauge fields,  $A, \bar{A} = \omega \pm \frac{1}{\ell} e$ ,

$$\begin{aligned} F = dA + A \wedge A &= -\frac{8\pi}{k} CB, \\ \bar{F} = d\bar{A} + \bar{A} \wedge \bar{A} &= -\frac{8\pi}{k} BC \end{aligned} \quad (4.63)$$

and Bianchi identity  $dF + A \wedge F - F \wedge A = 0$  are consistent with the equations of motion for the matter,  $dC + AC - CA = 0$  and  $dB - B \wedge A + \bar{A} \wedge B = 0$ . These equations describe the back reaction of the matter to the gravity. If the solution satisfies  $[B_{\mu\nu}, C] = 0$ , the torsion tensor (4.21) vanishes. Even in that case, gravity and the matter are still coupled. In general, the torsion does not vanish. In this case, as discussed before, the matter fields appear in the local translation of  $g_{\mu\nu}$  and  $\phi_{\mu\nu\lambda}$ , and the symmetry is enlarged compared to the pure spin-3 gravity case. Then the symmetry algebra of CFT on the boundary will be modified and the correlation functions in the boundary CFT will be affected. It will be worth finding black hole solutions and studying on the duality of this spin-3 gravity theory.

## 5 Summary

We expressed the generalized connections  $\Gamma_{\mu N}^M$  and curvature tensors  $R^M_{N\mu\nu}$ , which were introduced in our previous work[29], in terms of the metric  $g_{\mu\nu}$  and the spin-3 field  $\phi_{\mu\nu\lambda}$  explicitly. The matter coupling to 0-form  $C$  and 2-form  $B$  fields of the spin-3 gravity is introduced in the action formalism, firstly in the frame-like approach, and then translated into the metric-like approach. For this purpose, we eliminated the spin connection  $\omega_\mu^a$  by using the equations of motion. There are two options: one is to use the spin connection  $\omega_\mu^a(e)$  obtained by solving the equation of motion for the Chern-Simons theory representing the pure spin-3 gravity. The other is to solve the equation of motion for the total action to obtain  $\tilde{\omega}_\mu^a(e, B, C)$ . We adopted the second one, because the first one lead to an action which is not invariant under spin-3 transformation. After elimination of the spin connection,

interaction terms of the form  $(BC)^2$  are introduced due to the torsion. We showed that the symmetry is enhanced, when the matter fields are coupled to the spin-3 gravity. It was found that in order to construct spin-3 gravity theory, which is interacting with matter and is invariant under diffeomorphism, we need to consider topological field theory, like Chern-Simons theory and the BC system (4.3), in the frame-like approach.

Although we have not pursued further, the first option above for eliminating the spin connection might as well work. In this case it is necessary to modify the matter action and the transformation rule of the matter fields in order to recover the spin-3 gauge symmetry. This must be done by trial and error. If one could succeed in this, this would be more natural, because the transformation rules of  $g_{\mu\nu}$  and  $\phi_{\mu\nu\lambda}$  are not affected by the matters. In this case the size of the symmetry of the matter-coupled theory is the same as that of the pure spin-3 gravity. This attempt will be left for future study.

The construction of the matter coupling presented in this paper can be applied to other 3D higher-spin gravity theories based on  $SL(N, R) \times SL(N, R)$  and  $hs[\lambda] \times hs[\lambda]$  Chern-Simons theories, and a similar conclusion is expected. It is also possible to introduce topological matter composed of two 1-forms,  $C_\mu$  and  $B_\mu$  in a similar way.

## A Metric $G_{MN} = e_M^a e_{aN}$ and its inverse $G^{MN}$

$$G_{\mu\nu} = g_{\mu\nu}, \quad (\text{A.1})$$

$$G_{\mu(\nu\lambda)} = G_{(\nu\lambda)\mu} = \phi_{\mu\nu\lambda} - \frac{1}{3} g_{\nu\lambda} \phi_\mu, \quad (\text{A.2})$$

$$\begin{aligned} G_{(\mu\nu)(\lambda\rho)} &= M_{(\mu\nu)(\lambda\rho)} + (\phi_{\mu\nu\sigma} - \frac{1}{3} g_{\mu\nu} \phi_\sigma) g^{\sigma\kappa} (\phi_{\kappa\lambda\rho} - \frac{1}{3} g_{\lambda\rho} \phi_\kappa) \\ &= \frac{1}{4} g_{\mu\rho} g_{\nu\lambda} + \frac{1}{4} g_{\mu\lambda} g_{\nu\rho} - \frac{1}{6} g_{\mu\nu} g_{\lambda\rho} + \mathcal{O}(\phi^2), \end{aligned} \quad (\text{A.3})$$

$$G^{\mu\nu} = g^{\mu\nu} + 2 \phi^{\mu\lambda\rho} \phi^\nu_{\lambda\rho} - \frac{2}{3} \phi^\mu \phi^\nu + \mathcal{O}(\phi^4), \quad (\text{A.4})$$

$$G^{\mu(\nu\lambda)} = G^{(\nu\lambda)\mu} = -4\phi^{\mu\nu\lambda} + \frac{4}{3} g^{\nu\lambda} \phi^\mu + \mathcal{O}(\phi^3), \quad (\text{A.5})$$

$$G^{(\mu\nu)(\lambda\rho)} = J^{(\mu\nu)(\lambda\rho)} = 4 g^{\mu\rho} g^{\nu\lambda} + 4 g^{\mu\lambda} g^{\nu\rho} - \frac{8}{3} g^{\mu\nu} g^{\lambda\rho} + \mathcal{O}(\phi^2) \quad (\text{A.6})$$

## B Metric-like Tensor $K^M_{NK} = d^a_{bc} E^M_a e^b_N e^c_K$

$$K^\mu_{\nu\lambda} = \frac{2}{3} g_{\nu\lambda} \phi^\mu + \mathcal{O}(\phi^3), \quad (\text{B.1})$$

$$\begin{aligned} K^{(\mu\nu)}_{\lambda\rho} &= 4\delta^{(\mu}_\rho \delta^{\nu)}_\lambda - \frac{4}{3} g^{\mu\nu} g_{\lambda\rho} - \frac{8}{9} g_{\lambda\rho} \phi^{\mu\sigma\kappa} \phi^\nu_{\sigma\kappa} - \frac{4}{9} g_{\lambda\rho} \phi^\mu \phi^\nu \\ &\quad - \frac{8}{9} g_{\lambda\rho} \phi^{\mu\nu\sigma} \phi_\sigma + \frac{8}{27} g^{\mu\nu} g_{\lambda\rho} \phi_{\sigma\kappa\tau} \phi^{\sigma\kappa\tau} + \frac{4}{9} g^{\mu\nu} g_{\lambda\rho} \phi^\kappa \phi_\kappa + \mathcal{O}(\phi^4), \end{aligned} \quad (\text{B.2})$$

$$\begin{aligned} K^\mu_{(\nu\lambda)\rho} &= -\frac{1}{3} \delta^\mu_\rho g_{\nu\lambda} + \delta^\mu_{(\nu} g_{\lambda)\rho} - \frac{2}{3} \phi^\mu_{\rho}{}^\sigma \phi_{\nu\lambda\sigma} + \frac{2}{3} \phi^\mu_{(\nu}{}^\sigma \phi_{\lambda)\rho\sigma} + \frac{2}{3} g_{\rho(\nu} \phi_{\lambda)\sigma\kappa} \phi^{\mu\sigma\kappa} \\ &\quad + \frac{2}{3} \phi^\mu_{\rho(\nu} \phi_{\lambda)} - \frac{1}{3} g_{\rho(\nu} \phi^\mu_{\lambda)} - \frac{1}{3} \delta^\mu_{\rho} \phi_{\nu} \phi_{\lambda} - \frac{4}{9} g_{\nu\lambda} \phi^{\mu\sigma\kappa} \phi_{\rho\sigma\kappa} - \frac{2}{3} \phi^\mu_{\nu\lambda} \phi_\rho \\ &\quad + \frac{2}{9} g_{\nu\lambda} \phi^\mu_{\rho} \phi_\rho + \frac{1}{3} \delta^\mu_{(\nu} \phi_{\lambda)} \phi_\rho + \frac{2}{9} g_{\nu\lambda} \phi^\mu_{\rho}{}^\sigma \phi_\sigma + \frac{2}{3} \delta^\mu_{\rho} \phi_{\nu\lambda}{}^\sigma \phi_\sigma - \frac{2}{3} \delta^\mu_{(\nu} \phi_{\lambda)\rho}{}^\sigma \phi_\sigma \\ &\quad + \frac{1}{27} \delta^\mu_{\rho} g_{\nu\lambda} \phi_{\sigma\kappa\tau} \phi^{\sigma\kappa\tau} - \frac{1}{9} \delta^\mu_{(\nu} g_{\lambda)\rho} \phi_{\sigma\kappa\tau} \phi^{\sigma\kappa\tau} - \frac{1}{9} \delta^\mu_{\rho} g_{\nu\lambda} \phi^\kappa \phi_\kappa + \mathcal{O}(\phi^4), \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} K^{(\mu\nu)}_{(\lambda\rho)\sigma} &= -\frac{4}{3} g_{\sigma(\lambda} \phi^{\mu\nu}_{\rho)} + \frac{4}{3} g_{\lambda\rho} \phi^{\mu\nu}_\sigma + \frac{8}{3} \delta^{(\mu}_\sigma \phi^{\nu)}_{\lambda\rho} - \frac{8}{3} \delta^{(\mu}_{(\lambda} \phi^{\nu)}_{\rho)\sigma} - \frac{4}{3} \delta^{(\mu}_{(\lambda} g_{\rho)\sigma} \phi^{\nu)} \\ &\quad - \frac{4}{3} \delta^{(\mu}_\sigma \delta^{\nu)}_{(\lambda} \phi_{\rho)} + \frac{4}{3} g^{\mu\nu} g_{\sigma(\lambda} \phi_{\rho)} + \frac{4}{3} \delta^{(\mu}_{(\lambda} \delta^{\nu)}_{\rho)} \phi_\sigma - \frac{8}{9} g^{\mu\nu} g_{\lambda\rho} \phi_\sigma \\ &\quad + \mathcal{O}(\phi^3), \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned}
K^\mu_{(\nu\lambda)(\rho\sigma)} &= -\frac{2}{3}g_{\rho\sigma}\phi^\mu_{\nu\lambda} + \frac{2}{3}g_{\lambda(\rho}\phi^\mu_{\sigma)\nu} + \frac{2}{3}g_{\nu(\rho}\phi^\mu_{\sigma)\lambda} - \frac{2}{3}g_{\nu\lambda}\phi^\mu_{\rho\sigma} - \frac{1}{3}g_{\nu(\rho}g_{\sigma)\lambda}\phi^\mu \\
&\quad + \frac{1}{3}g_{\nu\lambda}g_{\rho\sigma}\phi^\mu + \frac{2}{3}\delta^\mu_{(\rho}\phi_{\sigma)\nu\lambda} + \frac{2}{3}\delta^\mu_{(\nu}\phi_{\lambda)\rho\sigma} - \frac{1}{3}\delta^\mu_{(\rho}g_{\sigma)(\nu}\phi_{\lambda)} - \frac{1}{3}\delta^\mu_{(\nu}g_{\lambda)(\rho}\phi_{\sigma)} \\
&\quad + \mathcal{O}(\phi^3), \tag{B.5} \\
K^{(\mu\nu)}_{(\lambda\rho)(\sigma\kappa)} &= \frac{4}{3}\delta^\mu_{(\kappa}\delta^\nu_{\sigma)}g_{\lambda\rho} + \frac{4}{3}\delta^\mu_{(\lambda}\delta^\nu_{\rho)}g_{\sigma\kappa} - \delta^\mu_{(\kappa}\delta^\nu_{\rho)}g_{\lambda\sigma} - \delta^\mu_{(\sigma}\delta^\nu_{\rho)}g_{\lambda\kappa} - \delta^\mu_{(\kappa}\delta^\nu_{\lambda)}g_{\rho\sigma} \\
&\quad - \delta^\mu_{(\sigma}\delta^\nu_{\lambda)}g_{\rho\kappa} + \frac{4}{3}g^{\mu\nu}g_{\lambda(\kappa}g_{\sigma)\rho} - \frac{8}{9}g^{\mu\nu}g_{\lambda\rho}g_{\sigma\kappa} + \mathcal{O}(\phi^2). \tag{B.6}
\end{aligned}$$

## C Generalized Curvature $R^M_{N\mu\nu}$

$$\begin{aligned}
R^{(\mu\nu)}_{(\lambda\rho)\sigma\kappa} &= 4\delta^{(\mu}_{(\lambda}\hat{R}^{\nu)}_{\rho)\sigma\kappa} + 2g^{\mu\nu}\left(\frac{2}{3}g_{\tau(\lambda|\hat{\Gamma}^\tau_{\kappa\eta}\hat{\Gamma}^\eta_{|\rho)\sigma} - \frac{2}{3}g_{\tau(\lambda|\hat{\Gamma}^\tau_{\sigma\eta}\hat{\Gamma}^\eta_{|\rho)\kappa}\right. \\
&\quad \left. + \frac{4}{9}g_{\tau(\lambda}\hat{\Gamma}^\tau_{\rho)\kappa}\hat{\Gamma}^\eta_{\sigma\eta} - \frac{4}{9}g_{\tau(\lambda}\hat{\Gamma}^\tau_{\rho)\sigma}\hat{\Gamma}^\eta_{\kappa\eta}\right) + \mathcal{O}(\phi^2), \tag{C.1}
\end{aligned}$$

In the above component, the  $\mathcal{O}(\phi^2)$  terms are not displayed.

$$\begin{aligned}
R^\mu_{(\nu\lambda)\rho\sigma} &= -\frac{1}{3}\hat{\nabla}_\rho\hat{\nabla}^\mu\phi_{\nu\lambda\sigma} - \frac{1}{3}g_{\sigma(\nu|\hat{\nabla}_\rho\hat{\nabla}^\mu\phi_{|\lambda)} + \frac{1}{3}\hat{\nabla}_\rho\hat{\nabla}_{(\nu}\phi^\mu_{\lambda)\sigma} - \frac{1}{3}g_{\sigma(\nu|\hat{\nabla}_\rho\hat{\nabla}_{|\lambda)}\phi^\mu \\
&\quad + \frac{2}{3}\delta^\mu_{\sigma}\hat{\nabla}_\rho\hat{\nabla}_{(\nu}\phi_{\lambda)} - \frac{1}{3}\delta^\mu_{(\nu|\hat{\nabla}_\rho\hat{\nabla}_{|\lambda)}\phi_\sigma + \frac{2}{3}\hat{\nabla}_\rho\hat{\nabla}_\sigma\phi^\mu_{\nu\lambda} - \frac{1}{3}\delta^\mu_{(\nu|\hat{\nabla}_\rho\hat{\nabla}_\sigma\phi_{|\lambda)} \\
&\quad + \frac{1}{3}g_{\sigma(\nu|\hat{\nabla}_\rho\hat{\nabla}_\kappa\phi_{|\lambda)}^{\kappa\mu} - \frac{1}{3}\delta^\mu_{\sigma}\hat{\nabla}_\rho\hat{\nabla}_\kappa\phi^\kappa_{\nu\lambda} + \frac{1}{3}\delta^\mu_{(\nu|\hat{\nabla}_\rho\hat{\nabla}_\kappa\phi_{|\lambda)\sigma}^\kappa \\
&\quad + \frac{1}{6}\delta^\mu_{(\nu}g_{\lambda)\sigma}\hat{\nabla}_\rho\hat{\nabla}_\kappa\phi^\kappa + \frac{1}{3}\hat{\nabla}_\sigma\hat{\nabla}^\mu\phi_{\nu\lambda\rho} + \frac{1}{3}g_{\rho(\nu|\hat{\nabla}_\sigma\hat{\nabla}^\mu\phi_{|\lambda)} - \frac{1}{3}\hat{\nabla}_\sigma\hat{\nabla}_{(\nu}\phi^\mu_{\lambda)\rho} \\
&\quad + \frac{1}{3}g_{\rho(\nu|\hat{\nabla}_\sigma\hat{\nabla}_{|\lambda)}\phi^\mu - \frac{2}{3}\delta^\mu_{\rho}\hat{\nabla}_\sigma\hat{\nabla}_{(\nu}\phi_{\lambda)} + \frac{1}{3}\delta^\mu_{(\nu|\hat{\nabla}_\sigma\hat{\nabla}_{|\lambda)}\phi_\rho - \frac{2}{3}\hat{\nabla}_\sigma\hat{\nabla}_\rho\phi^\mu_{\nu\lambda} \\
&\quad + \frac{1}{3}\delta^\mu_{(\nu|\hat{\nabla}_\sigma\hat{\nabla}_\rho\phi_{|\lambda)} - \frac{1}{3}g_{\rho(\nu|\hat{\nabla}_\sigma\hat{\nabla}_\kappa\phi^{\mu\kappa}_{|\lambda)} + \frac{1}{3}\delta^\mu_{\rho}\hat{\nabla}_\sigma\hat{\nabla}_\kappa\phi^{\kappa\mu}_{\nu\lambda} - \frac{1}{3}\delta^\mu_{(\nu|\hat{\nabla}_\sigma\hat{\nabla}_\kappa\phi_{|\lambda)\kappa\rho} \\
&\quad - \frac{1}{6}\delta^\mu_{(\nu}g_{\lambda)\rho}\hat{\nabla}_\sigma\hat{\nabla}_\kappa\phi^\kappa + g_{\nu\lambda}\left(\frac{1}{3}\hat{\nabla}_\rho\hat{\nabla}^\mu\phi_\sigma - \frac{1}{3}\hat{\nabla}_\rho\hat{\nabla}_\kappa\phi^{\kappa\mu}_\sigma - \frac{1}{6}\delta^\mu_{\sigma}\hat{\nabla}_\rho\hat{\nabla}_\kappa\phi^\kappa\right. \\
&\quad \left. - \frac{1}{3}\hat{\nabla}_\sigma\hat{\nabla}^\mu\phi_\rho + \frac{1}{3}\hat{\nabla}_\sigma\hat{\nabla}_\kappa\phi^{\kappa\mu}_\rho + \frac{1}{6}\delta^\mu_{\rho}\hat{\nabla}_\sigma\hat{\nabla}_\kappa\phi^\kappa\right) + \mathcal{O}(\phi^3), \tag{C.2}
\end{aligned}$$

$$\begin{aligned}
R^{(\mu\nu)}_{\lambda\rho\sigma} = & -\frac{4}{3}\hat{\nabla}_\rho\hat{\nabla}^{(\mu}\phi^{\nu)}_{\lambda\sigma} - \frac{8}{3}g_{\lambda\sigma}\hat{\nabla}_\rho\hat{\nabla}^{(\mu}\phi^{\nu)} + \frac{4}{3}\delta_\sigma^{(\mu}\hat{\nabla}_\rho\hat{\nabla}^{\nu)}\phi_\lambda + \frac{4}{3}\delta_\lambda^{(\mu}\hat{\nabla}_\rho\hat{\nabla}^{\nu)}\phi_\sigma \\
& + \frac{4}{3}\hat{\nabla}_\rho\hat{\nabla}_\lambda\phi^{\mu\nu}_\sigma + \frac{4}{3}\delta_\sigma^{(\mu}\hat{\nabla}_\rho\hat{\nabla}_\lambda\phi^{\nu)} + \frac{4}{3}\hat{\nabla}_\rho\hat{\nabla}_\sigma\phi^{\mu\nu}_\lambda + \frac{4}{3}\delta_\lambda^{(\mu}\hat{\nabla}_\rho\hat{\nabla}_\sigma\phi^{\nu)} \\
& + \frac{4}{3}g_{\lambda\sigma}\hat{\nabla}_\rho\hat{\nabla}_\kappa\phi^{\kappa\mu\nu} - \frac{4}{3}\delta_\sigma^{(\mu}\hat{\nabla}_\rho\hat{\nabla}_\kappa\phi^{\nu)\kappa}_\lambda - \frac{4}{3}\delta_\lambda^{(\mu}\hat{\nabla}_\rho\hat{\nabla}_\kappa\phi^{\nu)\kappa}_\sigma - \frac{2}{3}\delta_\sigma^{(\mu}\delta_\lambda^{\nu)}\hat{\nabla}_\rho\hat{\nabla}^\kappa\phi_\kappa \\
& + g^{\mu\nu}\left(-\frac{4}{3}\hat{\nabla}_\rho\hat{\nabla}_\lambda\phi_\sigma - \frac{4}{3}\hat{\nabla}_\rho\hat{\nabla}_\sigma\phi_\lambda + \frac{4}{3}\hat{\nabla}_\rho\hat{\nabla}^\kappa\phi_{\kappa\lambda\sigma} + \frac{2}{3}g_{\lambda\sigma}\hat{\nabla}_\rho\hat{\nabla}^\kappa\phi_\kappa\right. \\
& + \frac{8}{9}\hat{\Gamma}_{\sigma\tau}^\kappa\hat{\nabla}_\lambda\phi_{\rho\kappa}{}^\tau + \frac{8}{9}\hat{\Gamma}_{\rho\kappa}^\kappa\hat{\nabla}_\lambda\phi_\sigma + \frac{4}{9}\hat{\Gamma}_{\sigma\tau}^\kappa g_{\kappa\rho}\hat{\nabla}_\lambda\phi^\tau + \frac{8}{9}\hat{\Gamma}_{\sigma\tau}^\kappa\hat{\nabla}_\rho\phi_{\lambda\kappa}{}^\tau \\
& - \frac{8}{9}\hat{\Gamma}_{\sigma\kappa}^\kappa\hat{\nabla}_\rho\phi_\lambda + \frac{4}{9}\hat{\Gamma}_{\sigma\tau}^\kappa g_{\kappa\lambda}\hat{\nabla}_\rho\phi^\tau + \frac{4}{9}\hat{\Gamma}_{\lambda\sigma}^\kappa\hat{\nabla}_\rho\phi_\kappa - \frac{4}{9}\hat{\Gamma}_{\sigma\tau}^\kappa\hat{\nabla}_\kappa\phi^\tau{}_{\lambda\rho} \\
& - \frac{4}{9}\hat{\Gamma}_{\rho\tau}^\kappa g_{\kappa\sigma}\hat{\nabla}^\tau\phi_\lambda - \frac{4}{9}\hat{\Gamma}_{\rho\tau}^\kappa g_{\kappa\lambda}\hat{\nabla}^\tau\phi_\sigma - \frac{4}{9}\hat{\Gamma}_{\lambda\rho}^\kappa\hat{\nabla}_\kappa\phi_\sigma - \frac{8}{9}\hat{\Gamma}_{\sigma\tau}^\kappa g_{\lambda\rho}\hat{\nabla}_\kappa\phi^\tau \\
& - \frac{4}{9}\hat{\Gamma}_{\sigma\tau}^\kappa\hat{\nabla}^\tau\phi_{\kappa\lambda\rho} - \frac{8}{9}\hat{\Gamma}_{\rho\kappa}^\kappa\hat{\nabla}^\tau\phi_{\tau\lambda\sigma} - \frac{4}{9}\hat{\Gamma}_{\sigma\tau}^\kappa g_{\kappa\rho}\hat{\nabla}^\eta\phi_{\eta\lambda}{}^\tau - \frac{4}{9}\hat{\Gamma}_{\sigma\tau}^\kappa g_{\kappa\lambda}\hat{\nabla}^\eta\phi_{\eta\rho}{}^\tau \\
& + \frac{4}{9}\hat{\Gamma}_{\lambda\rho}^\kappa\hat{\nabla}^\tau\phi_{\kappa\tau\sigma} + \frac{8}{9}\hat{\Gamma}_{\rho\tau}^\kappa g_{\lambda\sigma}\hat{\nabla}^\tau\phi_\kappa + \frac{2}{9}\hat{\Gamma}_{\lambda\rho}^\kappa g_{\kappa\sigma}\hat{\nabla}^\tau\phi_\tau + \frac{8}{9}\hat{\Gamma}_{\sigma\tau}^\kappa g_{\lambda\rho}\hat{\nabla}^\eta\phi_{\eta\kappa}{}^\tau \\
& \left. + \frac{4}{9}\hat{\Gamma}_{\sigma\kappa}^\kappa g_{\lambda\rho}\hat{\nabla}^\tau\phi_\tau\right) - (\rho \leftrightarrow \sigma \text{ interchanged}) + \mathcal{O}(\phi^3). \tag{C.3}
\end{aligned}$$

The remaining component  $R^{(\mu\nu)}_{(\lambda\rho)\sigma\kappa}$  is too complicated and not presented.

## D Metric-Like Tensor $F_{\mu M}^N = f^a_{bc} e_\mu^c e_M^b E_a^N$

In the following equations

$$\epsilon^{\mu\nu\rho} = \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\rho}, \tag{D.1}$$

where  $\epsilon^{rt\phi} = +1$ . The indices are raised and lowered in terms of  $g_{\mu\nu}$  and  $g^{\mu\nu}$ .

$$\begin{aligned}
F_{\mu\nu}{}^\lambda = & \epsilon_{\mu\nu}{}^\lambda + \frac{2}{3}\epsilon^{\lambda\rho\sigma}\phi_{\mu\rho}{}^\kappa\phi_{\nu\sigma\kappa} + \frac{4}{3}\epsilon_{\mu\nu}{}^\rho\phi^{\lambda\sigma\kappa}\phi_{\rho\sigma\kappa} - \frac{4}{3}\epsilon_{\mu\nu}{}^\rho\phi^\lambda{}_\rho{}^\sigma\phi_\sigma \\
& + \frac{1}{6}\epsilon_{\mu\nu}{}^\lambda\phi^\sigma{}_\sigma\phi_\sigma + \mathcal{O}(\phi^4), \tag{D.2}
\end{aligned}$$

$$\begin{aligned}
F_{\mu(\nu\lambda)}{}^\rho = & \epsilon_\lambda{}^{\rho\sigma}\phi_{\mu\nu\sigma} + \epsilon_\nu{}^{\rho\sigma}\phi_{\mu\lambda\sigma} - 2\epsilon_\mu{}^{\rho\sigma}\phi_{\nu\lambda\sigma} - \frac{1}{2}\delta_\lambda{}^\rho\epsilon_{\mu\nu}{}^\sigma\phi_\sigma - \frac{1}{2}\delta_\nu{}^\rho\epsilon_{\mu\lambda}{}^\sigma\phi_\sigma \\
& - \frac{1}{2}\epsilon_\lambda{}^{\rho\sigma}g_{\mu\nu}\phi_\sigma - \frac{1}{2}\epsilon_\nu{}^{\rho\sigma}g_{\mu\lambda}\phi_\sigma + \frac{4}{3}\epsilon_\mu{}^{\rho\sigma}g_{\nu\lambda}\phi_\sigma + \mathcal{O}(\phi^3), \tag{D.3}
\end{aligned}$$

$$F_{\mu\nu}{}^{(\lambda\rho)} = 2\epsilon_\nu{}^{\rho\sigma}\phi_\mu{}^\lambda{}_\sigma + 2\epsilon_\nu{}^{\lambda\sigma}\phi_\mu{}^\rho{}_\sigma - 2\epsilon_\mu{}^{\rho\sigma}\phi_\nu{}^\lambda{}_\sigma - 2\epsilon_\mu{}^{\lambda\sigma}\phi_\nu{}^\rho{}_\sigma + \mathcal{O}(\phi^3), \tag{D.4}$$

$$F_{\mu(\nu\lambda)}{}^{(\rho\sigma)} = \delta_\lambda{}^\sigma\epsilon_{\mu\nu}{}^\rho + \delta_\lambda{}^\rho\epsilon_{\mu\nu}{}^\sigma + \delta_\nu{}^\sigma\epsilon_{\mu\lambda}{}^\rho + \delta_\nu{}^\rho\epsilon_{\mu\lambda}{}^\sigma + \mathcal{O}(\phi^2). \tag{D.5}$$

$$F_{(\mu\nu)(\lambda\rho)}{}^\sigma = \frac{1}{4}\epsilon_{\nu\rho}{}^\sigma g_{\mu\lambda} + \frac{1}{4}\epsilon_{\nu\lambda}{}^\sigma g_{\mu\rho} + \frac{1}{4}\epsilon_{\mu\rho}{}^\sigma g_{\nu\lambda} + \frac{1}{4}\epsilon_{\mu\lambda}{}^\sigma g_{\nu\rho} + \mathcal{O}(\phi^2) \tag{D.6}$$

$$\begin{aligned}
F_{(\mu\nu)(\lambda\rho)}^{(\kappa\sigma)} = & \frac{4}{3}\delta_{(\lambda|\varepsilon_\mu}^{(\kappa}\sigma)^{\tau}\phi_{\nu|\rho)\tau} + \frac{4}{3}\delta_{(\lambda|\varepsilon_\nu}^{(\kappa}\sigma)^{\tau}\phi_{\mu|\rho)\tau} - \frac{4}{3}\delta_{\mu}^{(\kappa}\varepsilon_{(\lambda|\sigma)^{\tau}\phi_{\nu|\rho)\tau} \\
& - \frac{4}{3}\delta_{\nu}^{(\kappa}\varepsilon_{(\lambda|\sigma)^{\tau}\phi_{\mu|\rho)\tau} - \frac{8}{3}\delta_{(\lambda\varepsilon_\rho)}^{(\kappa}\sigma)^{\tau}(\mu\phi_\nu)^{\sigma\tau} - \frac{8}{3}\varepsilon_{(\lambda}^{(\kappa|\tau}g_{\rho)(\mu}\phi_{\nu)}^{\sigma)^{\tau} \\
& + \frac{8}{3}\delta_{(\mu\varepsilon_\nu)}^{(\kappa}\sigma)^{\tau}(\lambda\phi_\rho)^{\sigma\tau} + \frac{8}{3}\varepsilon_{(\mu}^{\tau(\kappa}g_{\nu)(\lambda}\phi_\rho)^{\sigma)^{\tau} - 2\varepsilon_{\mu(\lambda}^{\tau}g_{\rho)\nu}\phi^{\kappa\sigma}_{\phantom{\kappa\sigma}\tau} \\
& - 2\varepsilon_{\nu(\lambda}^{\tau}g_{\rho)\mu}\phi^{\kappa\sigma}_{\phantom{\kappa\sigma}\tau} - 2\delta_{\mu}^{(\kappa}\delta_{(\lambda\varepsilon_\rho)}^{\sigma)\tau}\phi_\tau - 2\delta_{\nu}^{(\kappa}\delta_{(\lambda\varepsilon_\rho)}^{\sigma)\tau}\phi_\tau \\
& - 2\delta_{(\lambda|\varepsilon_\mu}^{(\kappa}\sigma)^{\tau}g_{\nu|\rho)\phi_\tau} - 2\delta_{(\lambda|\varepsilon_\nu}^{(\kappa}\sigma)^{\tau}g_{\mu|\rho)\phi_\tau} + 2\delta_{\mu}^{(\kappa}\varepsilon_{(\lambda}^{\sigma)\tau}g_{\rho)\nu}\phi_\tau \\
& + 2\delta_{\nu}^{(\kappa}\varepsilon_{(\lambda}^{\sigma)\tau}g_{\rho)\mu}\phi_\tau + \frac{4}{3}\delta_{(\lambda\varepsilon_\rho)}^{(\kappa}\sigma)^{\tau}g_{\mu\nu}\phi_\tau - \frac{4}{3}\delta_{(\mu\varepsilon_\nu)}^{(\kappa}\sigma)^{\tau}g_{\lambda\rho}\phi_\tau \\
& + \frac{4}{3}\varepsilon_{\mu(\lambda}^{\tau}g_{\rho)\nu}g^{\kappa\sigma}\phi_\tau + \frac{4}{3}\varepsilon_{\nu(\lambda}^{\tau}g_{\rho)\mu}g^{\kappa\sigma}\phi_\tau + \mathcal{O}(\phi^3)
\end{aligned} \tag{D.7}$$

## E Matter action up to $\mathcal{O}(\phi^1)$

The matter action is composed of the torsion part  $S_{\text{matter}}^{(\text{torsion})}$ , which come from  $\Delta\Gamma_{\mu M}^N$ , and the other part, which are expanded according to the power of  $\phi$ .

$$S_{\text{matter}} = S_{\text{matter}}^{(0)} + S_{\text{matter}}^{(1)} + \cdots + S_{\text{matter}}^{(\text{torsion})}, \tag{E.1}$$

Here only the part which do not depend on the torsion is presented up to  $\mathcal{O}(\phi^1)$ .

$$\begin{aligned}
S_{\text{matter}}^{(0)} = & \int d^3x \sqrt{-g} \left[ B_\mu{}^\lambda (\hat{\nabla}_\lambda C^\mu + \frac{1}{2\ell} g_{\lambda\rho} C^{(\mu\rho)}) + \frac{1}{2} B_{(\mu\rho)}{}^\lambda (\hat{\nabla}_\lambda C^{(\mu\rho)} + \frac{4}{\ell} \delta^\rho{}_\lambda C^\mu) \right. \\
& \left. + \frac{2}{\ell} B_\lambda{}^\lambda C^0 + B_0{}^\lambda (\partial_\lambda C^0 + \frac{4}{3\ell} g_{\lambda\nu} C^\nu) \right],
\end{aligned} \tag{E.2}$$

$$\begin{aligned}
S_{\text{matter}}^{(1)} = & \int d^3x \sqrt{-g} \frac{2}{3\ell} B_0{}^\mu C^{(\nu\lambda)} \phi_{\mu\nu\lambda} + \int d^3x \sqrt{-g} \frac{2}{3\ell} B_\mu{}^\lambda g_{\lambda\rho} C^\rho \phi^\mu \\
& + \int d^3x \sqrt{-g} B_\mu{}^\lambda C^{(\rho\sigma)} \left[ -\frac{1}{6} \hat{\nabla}^\mu \phi_{\lambda\rho\sigma} - \frac{1}{6} g_{\lambda\rho} \hat{\nabla}^\mu \phi_\sigma + \frac{1}{3} \hat{\nabla}_\lambda \phi^\mu{}_{\rho\sigma} \right. \\
& - \delta_\rho^\mu \hat{\nabla}_\lambda \phi_\sigma - \frac{1}{6} g_{\lambda\rho} \hat{\nabla}_\sigma \phi^\mu - \frac{1}{6} \delta_\rho^\mu \hat{\nabla}_\sigma \phi_\lambda + \frac{1}{3} \delta_\lambda^\mu \hat{\nabla}_\rho \phi_\sigma + \frac{1}{6} \hat{\nabla}_\rho \phi^\mu{}_{\lambda\sigma} \\
& + \frac{1}{6} g_{\lambda\rho} \hat{\nabla}^\kappa \phi^\mu{}_{\sigma\kappa} + \frac{1}{6} \delta_\rho^\mu \hat{\nabla}^\kappa \phi_{\lambda\sigma\kappa} - \frac{1}{6} \delta_\lambda^\mu \hat{\nabla}^\kappa \phi_{\kappa\rho\sigma} + \frac{1}{12} \delta_\rho^\mu g_{\lambda\sigma} \hat{\nabla}^\kappa \phi_\kappa \left. \right] \\
& + \int d^3x \sqrt{-g} B_{(\mu\nu)}{}^\lambda C^{(\rho\sigma)} \left[ -\frac{1}{3\ell} g_{\lambda\rho} \phi^{\mu\nu}{}_\sigma - \frac{2}{3\ell} \delta_\rho^\mu \phi^\nu{}_{\lambda\sigma} + \frac{2}{3\ell} \delta_\lambda^\mu \phi^\nu{}_{\rho\sigma} \right. \\
& - \frac{1}{3\ell} \delta_\rho^\mu g_{\lambda\sigma} \phi^\nu + \frac{1}{3\ell} \delta_\rho^\mu \delta_\sigma^\nu \phi_\lambda - \frac{1}{3\ell} \delta_\lambda^\mu \delta_\rho^\nu \phi_\sigma \left. \right] \\
& + \int d^3x \sqrt{-g} B_{(\mu\nu)}{}^\lambda C^\rho \left[ \frac{2}{3} \hat{\nabla}_\rho \phi^{\mu\nu}{}_\lambda + \frac{2}{3} \delta_\lambda^\mu \hat{\nabla}_\rho \phi^\nu + \frac{2}{3} \delta_\rho^\mu \hat{\nabla}^\nu \phi_\lambda \right. \\
& - \frac{2}{3} \hat{\nabla}^\mu \phi^\nu{}_{\rho\lambda} + \frac{2}{3} \delta_\lambda^\mu \hat{\nabla}^\nu \phi_\rho + \frac{2}{3} \delta_\rho^\mu \hat{\nabla}_\lambda \phi^\nu - \frac{4}{3} g_{\lambda\rho} \hat{\nabla}^\mu \phi^\nu + \frac{2}{3} \hat{\nabla}_\lambda \phi^{\mu\nu}{}_\rho \\
& \left. - \frac{2}{3} \delta_\lambda^\mu \hat{\nabla}^\kappa \phi_{\kappa}{}^\nu{}_\rho - \frac{2}{3} \delta_\rho^\mu \hat{\nabla}^\kappa \phi^\nu{}_{\kappa\lambda} + \frac{2}{3} g_{\lambda\rho} \hat{\nabla}_\kappa \phi^{\kappa\mu\nu} - \frac{1}{3} \delta_\lambda^\nu \delta_\rho^\mu \hat{\nabla}^\kappa \phi_\kappa \right].
\end{aligned} \tag{E.3}$$

## F Spin-3 transformations of $C^\mu$ and $C^{(\mu\nu)}$

Here only the parts of  $\delta C^M$ , (4.57) and (4.58), which do not depend on the torsion are presented up to  $\mathcal{O}(\phi^1)$ .

$$[\delta C^\mu]_0 = -\frac{1}{2\ell} \xi^{(\mu\nu)} g_{\nu\rho} C^\rho - \frac{1}{4} g_{\lambda\nu} C^{(\nu\rho)} \hat{\nabla}_\rho \xi^{(\mu\lambda)}, \quad (\text{F.1})$$

$$\begin{aligned} [\delta C^\mu]_1 = & \frac{1}{12\ell} C^{(\nu\lambda)} \phi_\nu \xi^{(\mu\rho)} g_{\rho\lambda} - \frac{1}{6\ell} C^{(\nu\lambda)} \phi_{\nu\lambda\rho} \xi^{(\mu\rho)} + \frac{1}{12\ell} C^{(\nu\lambda)} \phi^\mu \xi^{(\rho\sigma)} g_{\nu\rho} g_{\lambda\sigma} \\ & + \frac{1}{12\ell} C^{(\mu\nu)} \phi_\lambda \xi^{(\lambda\rho)} g_{\nu\rho} - \frac{1}{3\ell} C^{(\nu\lambda)} \phi^\mu_{\nu\sigma} \xi^{(\sigma\kappa)} g_{\kappa\lambda} - \frac{1}{6\ell} C^{(\mu\nu)} \phi_{\nu\lambda\rho} \xi^{(\lambda\rho)} \\ & + \frac{1}{6} C^\nu \xi^{(\lambda\rho)} \hat{\nabla}^\mu \phi_{\nu\lambda\rho} + \frac{1}{6} C^\nu g_{\nu\lambda} \xi^{(\lambda\rho)} \hat{\nabla}^\mu \phi_\rho - \frac{1}{3} C^\nu \xi^{(\lambda\rho)} \hat{\nabla}_\nu \phi^\mu_{\lambda\rho} \\ & + \frac{1}{6} C^\nu \xi^{(\mu\lambda)} \hat{\nabla}_\nu \phi_\lambda + \frac{1}{6} C^\nu g_{\nu\lambda} \xi^{(\lambda\rho)} \hat{\nabla}_\rho \phi^\mu + \frac{1}{6} C^\nu \xi^{(\mu\lambda)} \hat{\nabla}_\lambda \phi_\nu \\ & - \frac{1}{3} C^\mu \xi^{(\nu\lambda)} \hat{\nabla}_\lambda \phi_\nu - \frac{1}{6} C^\nu \xi^{(\lambda\rho)} \hat{\nabla}_\rho \phi^\mu_{\nu\lambda} - \frac{1}{6} C^\nu g_{\nu\lambda} \xi^{(\lambda\rho)} \hat{\nabla}_\sigma \phi^\mu_{\rho}{}^\sigma \\ & - \frac{1}{6} C^\nu \xi^{(\mu\lambda)} \hat{\nabla}_\rho \phi_{\nu\lambda}{}^\rho + \frac{1}{6} C^\mu \xi^{(\nu\lambda)} \hat{\nabla}_\rho \phi_{\nu\lambda}{}^\rho - \frac{1}{12} C^\nu g_{\nu\lambda} \xi^{(\mu\lambda)} \hat{\nabla}_\rho \phi^\rho, \quad (\text{F.2}) \end{aligned}$$

$$\begin{aligned} [\delta C^{(\mu\nu)}]_0 = & -\frac{2}{\ell} C^0 \xi^{(\mu\nu)} + \frac{1}{2\ell} C^{(\mu\rho)} g_{\rho\lambda} \xi^{(\nu\lambda)} + \frac{1}{2\ell} C^{(\nu\rho)} g_{\rho\lambda} \xi^{(\mu\lambda)} \\ & - \frac{1}{3\ell} g^{\mu\nu} C^{(\lambda\rho)} \xi^{(\sigma\kappa)} g_{\lambda\sigma} g_{\rho\kappa} - C^\lambda \hat{\nabla}_\lambda \xi^{(\mu\nu)}, \quad (\text{F.3}) \end{aligned}$$

$$\begin{aligned} [\delta C^{(\mu\nu)}]_1 = & -\frac{1}{3\ell} \xi^{(\mu\nu)} C^\lambda \phi_\lambda + \frac{1}{3\ell} \xi^{(\mu\lambda)} g_{\lambda\rho} C^\rho \phi^\nu + \frac{1}{3\ell} \xi^{(\mu\lambda)} C^\nu \phi_\lambda \\ & + \frac{2}{3\ell} \xi^{(\mu\lambda)} C^\rho \phi^\nu_{\lambda\rho} + \frac{1}{3\ell} \phi^{\mu\nu}_{\lambda} C^\rho g_{\rho\sigma} \xi^{(\lambda\sigma)} - \frac{1}{3\ell} g^{\mu\nu} C^\lambda \xi^{(\rho\sigma)} g_{\sigma\lambda} \phi_\rho \\ & - \frac{2}{3\ell} C^\nu \phi^\mu_{\lambda\rho} \xi^{(\lambda\rho)} + \frac{1}{6} C^{(\nu\lambda)} \xi^{(\rho\sigma)} \hat{\nabla}^\mu \phi_{\lambda\rho\sigma} + \frac{1}{6} C^{(\nu\lambda)} g_{\lambda\rho} \xi^{(\rho\sigma)} \hat{\nabla}^\mu \phi_\sigma \\ & - \frac{1}{3} C^{(\nu\lambda)} \xi^{(\rho\sigma)} \hat{\nabla}_\lambda \phi^\mu_{\rho\sigma} + \frac{1}{6} C^{(\nu\lambda)} \xi^{(\mu\rho)} \hat{\nabla} \phi_\rho + \frac{1}{6} C^{(\nu\lambda)} \phi_\rho \hat{\nabla} \xi^{(\mu\rho)} \\ & - \frac{1}{3} C^{(\nu\lambda)} \phi^\mu_{\rho\sigma} \hat{\nabla} \xi^{(\rho\sigma)} + \frac{1}{6} C^{(\nu\lambda)} g_{\lambda\rho} \xi^{(\rho\sigma)} \hat{\nabla}_\sigma \phi^\mu \\ & + \frac{1}{18} g^{\mu\nu} C^{(\lambda\rho)} \xi^{(\sigma\kappa)} \hat{\nabla}_\rho \phi_{\lambda\sigma\kappa} - \frac{1}{9} g^{\mu\nu} C^{(\lambda\rho)} g_{\rho\sigma} \xi^{(\sigma\kappa)} \hat{\nabla}_\lambda \phi_\kappa + \frac{1}{6} C^{(\nu\lambda)} \xi^{(\mu\rho)} \hat{\nabla}_\rho \phi_\lambda \\ & - \frac{1}{3} C^{(\mu\nu)} \xi^{(\lambda\rho)} \hat{\nabla}_\rho \phi_\lambda - \frac{1}{6} C^{(\lambda\rho)} \phi_\lambda \hat{\nabla}_\rho \xi^{(\mu\nu)} + \frac{1}{6} C^{(\lambda\rho)} \phi^\nu \hat{\nabla}_\rho \xi^{(\mu\sigma)} g_{\sigma\lambda} \\ & + \frac{1}{3} C^{(\lambda\rho)} \phi^\nu_{\lambda\sigma} \hat{\nabla}_\rho \xi^{(\mu\sigma)} + \frac{1}{6} C^{(\lambda\rho)} g^{\mu\nu} \phi_\sigma \hat{\nabla}_\rho \xi^{(\kappa\sigma)} g_{\lambda\kappa} \\ & - \frac{1}{6} C^{(\lambda\rho)} g^{\mu\nu} \phi_\sigma \hat{\nabla}_\rho \xi^{(\kappa\sigma)} g_{\lambda\kappa} - \frac{1}{3} g^{\mu\nu} C^{(\lambda\rho)} \phi_{\lambda\sigma\kappa} \hat{\nabla}_\rho \xi^{(\sigma\kappa)} \\ & - \frac{1}{6} C^{(\nu\lambda)} \xi^{(\rho\sigma)} \hat{\nabla}_\sigma \phi^\mu_{\lambda\rho} - \frac{1}{6} C^{(\nu\lambda)} g_{\lambda\rho} \xi^{(\rho\sigma)} \hat{\nabla}^\kappa \phi^\mu_{\sigma\kappa} - \frac{1}{6} C^{(\nu\lambda)} \xi^{(\mu\rho)} \hat{\nabla}^\kappa \phi_{\lambda\rho\kappa} \\ & + \frac{1}{6} C^{(\mu\nu)} \xi^{(\lambda\rho)} \hat{\nabla}^\sigma \phi_{\lambda\rho\sigma} - \frac{1}{9} g^{\mu\nu} C^{(\lambda\rho)} g_{\lambda\sigma} \xi^{(\sigma\kappa)} \hat{\nabla}_\kappa \phi_\rho - \frac{1}{12} C^{(\nu\lambda)} g_{\lambda\rho} \hat{\nabla}_\kappa \phi^\kappa \\ & + \frac{1}{18} g^{\mu\nu} C^{(\lambda\rho)} \xi^{(\sigma\kappa)} \hat{\nabla}_\kappa \phi_{\lambda\rho\sigma} + \frac{1}{9} g^{\mu\nu} C^{(\lambda\rho)} g_{\lambda\sigma} \xi^{(\sigma\kappa)} \hat{\nabla}_\tau \phi_{\rho\kappa}{}^\tau \\ & + \frac{1}{36} C^{(\lambda\rho)} g^{\mu\nu} \xi^{(\sigma\kappa)} g_{\lambda\sigma} g_{\rho\kappa} \hat{\nabla}_\tau \phi^\tau + (\mu \leftrightarrow \nu \text{ interchanged}). \quad (\text{F.4}) \end{aligned}$$

For  $\delta C^0 = [\delta C^0]_0 + [\delta C^0]_1 + \mathcal{O}(\phi^2)$ , we have

$$\begin{aligned} [\delta C^0]_0 &= -\frac{1}{6\ell} \xi^{(\mu\nu)} C^{(\lambda\rho)} g_{\mu\lambda} g_{\nu\rho}, \\ [\delta C^0]_1 &= -\frac{2}{3\ell} \xi^{(\mu\nu)} C^\lambda \phi_{\mu\nu\lambda}. \end{aligned} \quad (\text{F.5})$$

## G $[\Delta \Gamma_{\mu\nu}^{(\tau\eta)}]_1$

$\mathcal{O}(\phi^1)$  corrections to  $\Delta \Gamma_{\mu\nu}^{(\tau\eta)}$  in (4.24) are presented in this appendix. Here the indices of  $T_{\mu\nu,\lambda}$  are raised by  $g^{\mu\nu}$ .

$$\begin{aligned} [\Delta \Gamma_{\mu\nu}^{(\tau\eta)}]_1 &= \frac{1}{8} J^{(\tau\eta)(\lambda\rho)} \left[ \frac{1}{3} g_{\nu\rho} \phi_\sigma T_{\mu\lambda}{}^{,\sigma} + \frac{1}{3} g_{\nu\lambda} \phi_\sigma T_{\mu\rho}{}^{,\sigma} - \phi_{\nu\rho\sigma} T_\mu{}^\sigma{}_{,\lambda} - \phi_{\nu\lambda\sigma} T_\mu{}^\sigma{}_{,\rho} \right. \\ &\quad + 2g_{\lambda\rho} \phi_{\nu\sigma\kappa} T_\mu{}^{\sigma,\kappa} - g_{\nu\rho} \phi_{\lambda\sigma\kappa} T_\mu{}^{\sigma,\kappa} - g_{\nu\lambda} \phi_{\rho\sigma\kappa} T_\mu{}^{\sigma,\kappa} + \frac{1}{3} g_{\mu\rho} \phi_\sigma T_{\nu\lambda}{}^{,\sigma} \\ &\quad + \frac{1}{3} g_{\mu\lambda} \phi_\sigma T_{\nu\rho}{}^{,\sigma} - \phi_{\mu\rho\sigma} T_\nu{}^\sigma{}_{,\lambda} - \phi_{\mu\lambda\sigma} T_\nu{}^\sigma{}_{,\rho} + 2g_{\lambda\rho} \phi_{\mu\sigma\kappa} T_\nu{}^{\sigma,\kappa} - g_{\mu\rho} \phi_{\lambda\sigma\kappa} T_\nu{}^{\sigma,\kappa} \\ &\quad - g_{\mu\lambda} \phi_{\rho\sigma\kappa} T_\nu{}^{\sigma,\kappa} - \phi_{\nu\rho\sigma} T_\lambda{}^\sigma{}_{,\mu} - \phi_{\mu\rho\sigma} T_\lambda{}^\sigma{}_{,\nu} + 2\phi_{\mu\nu\sigma} T_\lambda{}^\sigma{}_{,\rho} - g_{\nu\rho} \phi_{\mu\sigma\kappa} T_\lambda{}^{\sigma,\kappa} \\ &\quad - g_{\mu\rho} \phi_{\nu\sigma\kappa} T_\lambda{}^{\sigma,\kappa} + 2g_{\mu\nu} \phi_{\rho\sigma\kappa} T_\lambda{}^{\sigma,\kappa} - \phi_{\nu\lambda\sigma} T_\rho{}^\sigma{}_{,\mu} - \phi_{\mu\lambda\sigma} T_\rho{}^\sigma{}_{,\nu} + 2\phi_{\mu\nu\sigma} T_\rho{}^\sigma{}_{,\lambda} \\ &\quad - g_{\nu\lambda} \phi_{\mu\sigma\kappa} T_\rho{}^{\sigma,\kappa} - g_{\mu\lambda} \phi_{\nu\sigma\kappa} T_\rho{}^{\sigma,\kappa} + 2g_{\mu\nu} \phi_{\lambda\sigma\kappa} T_\rho{}^{\sigma,\kappa} - 4g_{\lambda\rho} \phi_{\mu\nu\kappa} T^{\sigma\kappa}{}_{,\sigma} \\ &\quad + 2g_{\nu\rho} \phi_{\mu\lambda\kappa} T^{\sigma\kappa}{}_{,\sigma} + 2g_{\nu\lambda} \phi_{\mu\rho\kappa} T^{\sigma\kappa}{}_{,\sigma} + 2g_{\mu\rho} \phi_{\nu\lambda\kappa} T^{\sigma\kappa}{}_{,\sigma} + 2g_{\mu\lambda} \phi_{\nu\rho\kappa} T^{\sigma\kappa}{}_{,\sigma} \\ &\quad \left. - 4g_{\mu\nu} \phi_{\lambda\rho\kappa} T^{\sigma\kappa}{}_{,\sigma} - g_{\mu\rho} g_{\nu\lambda} \phi_\kappa{}^\tau T^{\sigma\kappa}{}_{,\sigma} - g_{\mu\lambda} g_{\nu\rho} \phi_\kappa{}^\tau T^{\sigma\kappa}{}_{,\sigma} + 2g_{\mu\nu} g_{\lambda\rho} \phi_\kappa{}^\tau T^{\sigma\kappa}{}_{,\sigma} \right] \end{aligned} \quad (\text{G.1})$$

## References

- [1] E. S. Fradkin and M. A. Vasiliev, *On the gravitational interaction of massless higher spin fields*, Phys. Lett. B 189 (1987) 89.
- [2] E. S. Fradkin and M. A. Vasiliev, *Candidate for the role of higher-spin gravity*, Ann. Phys. 177 (1987) 63.
- [3] M. A. Vasiliev, *Progress in higher spin gauge theories*, [arXiv:hep-th/0104246].
- [4] M. P. Blencowe, *A consistent interacting massless higher-spin field theory in  $D=2+1$* , Class. Quantum Grav. 6 (1989) 443-452.
- [5] M. R. Gaberdiel and R. Gopakumar, *An  $AdS_3$  dual for minimal model CFTs*, [arXiv:1011.2986 [hep-th]].
- [6] M. R. Gaberdiel and T. Hartman, *Symmetries of holographic minimal models*, [arXiv:1101.2910 [hep-th]].



- [7] C. Ahn, *The large  $N$  't Hooft limit of coset minimal models*, [arXiv: 1106.0351 [hep-th]].
- [8] M. R. Gaberdiel, R. Gopakumar, T. Hartman and S. Raju, *Partition functions of holographic minimal models*, [arXiv: 1106.1897 [hep-th]].
- [9] M. R. Gaberdiel and R. Gopakumar, *Minimal Model Holography*, [arXiv:hep-th/1207.6697].
- [10] C-M. Chang and X. Yin, *Higher spin gravity with matter in  $AdS_3$  and its CFT dual*, [arXiv:1106.2580[hep-th]].
- [11] P. Kraus and E. Perlmutter, *Probing higher spin black holes*, [arXiv:hep-th/1209.4937].
- [12] M. Ammon, P. Kraus and E. Perlmutter, *Scalar fields and three-point functions in  $D = 3$  higher spin gravity*, [arXiv:hep-th/1111.3926].
- [13] E. Hijano, P. Kraus and E. Perlmutter, *Matching four-point functions in higher spin  $AdS_3/CFT_2$* , [arXiv:hep-th/1302.6113].
- [14] C. Ahn, *The higher spin currents in the  $N=1$  stringy coset minimal model*, JHEP 04 (2013) 033, [arXiv:1211.2589 [hep-th]].
- [15] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, [arXiv:1008.4744 [hep-th]].
- [16] A. Campoleoni, *Higher spins in  $D=2+1$* , [arXiv:1110.5841 [hep-th]].
- [17] M. Gutperle and P. Kraus, *Higher spin black hole*, [arXiv:1103.4304 [hep-th]].
- [18] M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, *Spacetime geometry in higher spin gravity*, [arXiv:1106.4788 [hep-th]].
- [19] P. Kraus and E. Perlmutter, *Partition functions of higher spin black holes and their CFT duals*, [arXiv:1108.2567 [hep-th]].
- [20] M. R. Gabardiel, T. Hartman and K. Jin, *Higher spin black holes from CFT*, [arXiv:1203.0015 [hep-th]].
- [21] M. Bañados, R. Canto and S. Theisen, *The action for higher spin black holes in three dimensions*, [arXiv: 1204.5105 [hep-th]].
- [22] M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, *Black holes in three dimensional higher spin gravity: a review*, [arXiv:1208.5182 [hep-th]].

- [23] B. Chen, J. Long and Y-N. Wang, *Black holes in truncated Higher spin AdS3 gravity*, JHEP 1212 (2012) 052, [arXiv:1209.6185 [hep-th]].
- [24] B. Chen, J. Long and Y-N. Wang, *D2 Chern-Simons gravity*, [arXiv:1211.6917 [hep-th]].
- [25] C. Fronsdal, *Massless fields with integer spin*, Phys. Rev. **D18** (1978) 3624; J. Fang and C. Fronsdal, *Massless fields with half-integer spin*, Phys. Rev. **D18** (1978) 3630.
- [26] S. Lal and B. Sahoo, *Holographic renormalisation for the spin-3 theory and the (A)dS3/CFT2 correspondence*, JHEP 1301 (2013) 004, [arXiv:1209.4804 [hep-th]].
- [27] A. Fotopoulos and M. Tsulaia, *Gauge invariant Lagrangians for free and interacting higher spin fields. A review of the BRST formulation*, [arXiv:hep-th/0805.1346].
- [28] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Towards metric-like higher-spin gauge theories in three dimensions*, [ArXiv: 1208.1851 [hep-th]].
- [29] I. Fujisawa and R. Nakayama, *Second-Order Formalism for 3D Spin-3 Gravity*, Class. Quantum Grav. **30** (2013) 035003, [ArXiv: 1209.0894 [hep-th]].
- [30] J. M. Martin-Garcia, *xAct: Efficient tensor computer algebra for Mathematica*, <http://xact.es/>
- [31] G.M.T. Watts, *W-algebras and their representations*, Lectures given at the 1996 Eötvös Summer School on CFT and Integrable Models, Bolyai College, Budapest, KCL-MTH-97-50, <http://www.mth.kcl.ac.uk/gmtw/97-50.ps>.
- [32] A. Achúcarro and P. K. Townsend, *A Chern-Simons action for three-dimensional anti-de Sitter supergravity theories*, Phys. Lett. B180 (1986) 89.
- [33] E. Witten, *2+1 dimensional gravity as an exactly soluble system*, Nucl. Phys. B311 (1988) 46.
- [34] M. Henneaux and S.-J. Rey, *Nonlinear  $W_\infty$  as asymptotic symmetry of three-dimensional higher spin AdS gravity*, [arXiv:1008.4579 [hep-th]].
- [35] M. R. Gabadiel, R. Gopakumar and A. Saha, *Quantum W-symmetry in AdS3*, JHEP 1102 (2011) 004 [arXiv:1009.6087 [hep-th]].
- [36] B. de Wit and D. Z. Freedman, *Systematics of higher-spin gauge fields*, Phys. Rev. **D21** (1980) 358.

- [37] M. Henningson and K. Sfetsos, *Spinors and the AdS/CFT correspondence*, Phys. Lett. **B431** (1998) 63, [arXiv:hep-th/9803251].
- [38] J. M. Maldacena, *The large  $N$  limit of superconformal field theories and supergravity*, Adv. Theor. Phys. **2** (1998) 231, [arXiv:hep-th/9711200].
- [39] S. S. Gubser, I. R. Klebanov, A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. **B428** (1998) 105, [arXiv:hep-th/9802109].
- [40] E. Witten, *Anti de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253, [arXiv:hep-th/9802150].